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HSU-STRUCTURE ON DIFFERENTIABLE STRUCTURES DEFINED BY ALGEBRAIC EQUATIONS, I. NIJENHUIS TENSOR

Geeta Verma*

(Received 07-12-2004)

ABSTRACT

In differential geometry, one put an additional structure on the differentiable manifold (a vector field, a 2-form, a Riemannian metric) and studies properties connected with these geometric objects. One way of introducing additional structure is by means of a vector valued linear function F satisfying some algebraic relations. These structures have been studied extensively under various topics such as complex and almost complex spaces, almost product spaces, almost tangent spaces. Recently, R.S. Mishra [3] has developed a new approach by introducing the index-free notations which evidently helps in avoiding manipulation of indices and represents the entire subject in a more general way. The aim of the present paper is to follow this new approach with special emphasis on discussing above mentioned structures collectively so that more advanced expositions can be given without having to start from the very beginning.

Keywords: Algebraic equations, Nijenhuis tensor, Hsu-structure, Riemannian metric

PRELIMINARY

We consider a differentiable manifold V_n of class C^∞. Let there exists on V_n a vector valued linear function F of class C^∞ such that.

$$\overset{=}{x} = a^r x \tag{1.1}$$

for an arbitrary vector field x, where $\overset{-}{x} \overset{def}{=} F(x)$ and a is any complex number. Let us agree to say that F gives to V_n a differentiable structure, briefly Hsu-structure defined by the algebraic equation (1.1). It is well known that V_n is endowed with a π-structure [2] or an almost product structure or an almost comlex structure [3] or an almost tangent structure [1] according as $a \neq 0$ or $a = 1$ or $a = I$ or $a = 0$. The rank of F in the first three cases in n and in the last case it is $n/2$. It the last two cases n has to be even. If the given Hsu-structure is endowed with a Hermite metric g such that $g(\overset{-}{x}, \overset{-}{y}) + a^r g(x,y) = 0$, then we say that (F,g) gives to V_n a Hermite structure, briefly H-Structure, subordinate to Hsu-structure.

In the sequel, arbitrary vector fields will be denoted by $x, y, z, \ldots\ldots\ldots\ldots$, etc. Let us consider on V_n, equipped with H-Stucture, a tensor f of type (o,2) such that

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$$f(x,y) \overset{def}{=} g(\bar{x},y) = -g(x,\bar{y}) \tag{1.2a}$$

It is easy to verify the following results:

$$f(\bar{x},y) = -f(x,\bar{y}) = a^r g(x,y) \tag{1.2b}$$

$$f(\bar{x},\bar{y}) = -a^r g(\bar{x},y) = a^r g(x,\bar{y}) = -a^r f(x,y) \tag{1.2c}$$

Since g symmetric, equations (1.2a) and (1.2c) yield that f is skew - symmetric.

NIJENHUIS TENSOR

The Nijenhuis tensor with respect to F is a vector valued bilinear function N given by

$$N(x,y) \overset{def}{=} [\bar{x},\bar{y}] + \overline{\overline{[x,y]}} - \overline{[\bar{x},y]} - \overline{[\bar{x},y]} + S(\bar{x},\bar{y}) + \overline{\overline{S(x,y)}} \tag{2.1}$$

where S is the torsion tensor with respect to a connection D. One can easily verify

$$N(x,y) = -N(y,x) \tag{2.2}$$

Theorem 2.1

If V_n is equipped with a Hsu-structure, then

$$N(x,\bar{y}) = N(\bar{x},y) = -\overline{N(x,y)} \tag{2.3a}$$

$$N(\bar{x},\bar{y}) = a^r N(x,y) = -\overline{N(x,\bar{y})} = -\overline{N(\bar{x},y)} \tag{2.3b}$$

Proof: Using (1.1), we obtain the required results from the following equations:

$$N(x,y) = [\bar{x},\bar{y}] + a^r[x,y] - \overline{[x,\bar{y}]} - \overline{[\bar{x},y]} + S(\bar{x},\bar{y}) + a^r S(x,y) - \overline{S(x,\bar{y})} - \overline{S(\bar{x},y)}$$

$$\overline{N(x,y)} = [\bar{x},\bar{y}] + a^r\overline{[x,y]} - a^r[x,\bar{y}] - a^r[\bar{x},y] + \overline{S(\bar{x},\bar{y})} + a^r\overline{S(x,y)} - a^r S(x,\bar{y}) - a^r S(\bar{x},y)$$

$$\overline{N(\bar{x},y)} = a^r[\bar{x},y] + a^r\overline{[\bar{x},\bar{y}]} - a^r\overline{[\bar{x},y]} - a^r[\bar{x},\bar{y}] - a^{2r}[x,y] + a^r\overline{S(x,\bar{y})} + a^r\overline{S(\bar{x},y)} - a^r S(\bar{x},\bar{y})$$
$$- a^{2r}S(x,y) = \overline{N(x,\bar{y})}$$

$$N(\bar{x},\bar{y}) = a^{2r}[x,y] + a^r[\bar{x},\bar{y}] - a^r\overline{[\bar{x},y]} - a^r\overline{[x,\bar{y}]} + a^{2r}S(x,y) + a^r S(\bar{x},\bar{y}) - a^r\overline{S(\bar{x},y)} - a^r\overline{S(x,\bar{y})}$$
$$= a^r N(x,y) = -\overline{N(\bar{x},y)} = --\overline{N(x,\bar{y})}$$

Corollary : If V_n is equipped with an almost tangent structure, then

$$N(\bar{x},\bar{y}) = \overline{N(x,\bar{y})} = \overline{N(\bar{x},y)} = 0$$

Definition: A bilinear function ϕ is said to be pure in the two slots if $\phi(\bar{x},\bar{y}) - a^r\phi(x,y) = 0.$ It is said to be hybrid in the two slots if $\phi(\bar{x},\bar{y}) + a^r\phi(x,y) = 0.$

Proposition 1: In a Hsu - structure, the Nijenhuis tensor is pure in both the slots.

Proposition 2 : In an almost tangent structure, the Nijenhuis tensor is pure as well as hybrid in both the slots.

Theorem 2.2

Let V_n be equipped with a Hsu-structure. If we put

$$T(x,y) \overset{def}{=} [\bar{x},\bar{y}] + a^r[x,y] + S(\bar{x},\bar{y}) + a^r S(x,y) \tag{2.4a}$$

$$P(x,y) \overset{def}{=} [\bar{x},\bar{y}] - \overline{[x,\bar{y}]} + S(\bar{x},\bar{y}) - S(x,\bar{y}) \tag{2.4b}$$

then

$$N(\bar{x},\bar{y}) = a^r T(x,y) - \overline{T(x,\bar{y})} = a^r P(x,y) + P(\bar{x},\bar{y})$$

Moreover, $T(x,y)$ is skew-symmetric.

Proof: From (2.4) and (1.1), we have

$$\overline{T(x,\bar{y})} = a^r\overline{[x,\bar{y}]} + a^r\overline{[\bar{x},y]} + a^r\overline{S(\bar{x},y)} + a^r\overline{S(x,\bar{y})}$$

$$P(\bar{x},\bar{y}) = a^{2r}[x,y] - a^r\overline{[x,\bar{y}]} + a^{2r}S(x,y) - a^r\overline{S(x,\bar{y})}$$

Also from (2.3)b), (2.1) and (1.1), we get

$$N(\bar{x},\bar{y}) = a^r N(x,y)$$

$$= a^r\left\{[\bar{x},\bar{y}] + a^r[x,y] + S(\bar{x},\bar{y}) + a^r S(x,y)\right\} - a^r\left\{\overline{[\bar{x},y]} + \overline{[x,\bar{y}]} + \overline{S(\bar{x},y)} + \overline{S(x,\bar{y})}\right\}$$

$$= a^r T(x,y) - \overline{T(x,\bar{y})}$$

The remaining result can be proved similarly. The relation (2.4a) obviously implies that $T(x,y)$ is skew-symmetric.

Corollary 1: In an almost tangent structure, $P(\bar{x},\bar{y}) = \overline{T(x,\bar{y})} = 0$

Corollary 2 : In a Hsu-structure, $T(x,y)$ is pure in both the slots.

Let us suppose that V_n is equipped with an H-structure subordinate to Hsu-structure. If we identify the given connection D to a suitable connection with respect to g then the Nijenhuis Tensor for F will be given by

$$N(x,y) = [\bar{x},\bar{y}] + a^r[x,y] - \overline{[x,\bar{y}]} - \overline{[\bar{x},y]}$$

Furthermore, for an almost Hermite space, we have

$$N(x,y) = [\bar{x},\bar{y}] - [x,y] - \overline{[x,\bar{y}]} - \overline{[\bar{x},y]}$$

One can easily verify that the results of theorems (2.1) and (2.3) of R.S. Mishra [3] are particular cases of theorem (2.2) of this paper.

In the sequel we shall assume that V_n is euipped with an H-structure unless stated otherwise:

Theorem 2.3

Let us put

$$N(x,y,z) \overset{def}{=} -a^r g(N(x,y),z) = -g(N(\bar{x},\bar{y}),z) \tag{2.5a}$$

Then N (x,y,z) is skew symmetric in x,y.

$$N(x,y,z) = - N(y,x,z) \tag{2.5b}$$

$$N(\bar{x},\bar{y},z) = N(\bar{x},y,\bar{z}) = N(x,\bar{y},\bar{z}) = a^r N(x,y,z) \tag{2.5c}$$

Proof: Relation (2.5b) follows immediately from (2.5a) and (2.2). From (2.5a) and (1.1), we have

$$N(\bar{x},\bar{y},z) = -g(N(\bar{\bar{x}},\bar{\bar{y}}),z) = -a^{2r} g(N(x,y),z) = a^r N(x,y,z)$$

$$N(\bar{x},y,\bar{z}) = -g(N(\bar{\bar{x}},\bar{y}),z) = -a^r g(N(x,\bar{y}),z)$$

Using (1.2 a) and then (2.3 b), we get

$$N(\bar{x},\bar{y},z)=-a^r g(N(\bar{x},\bar{y}),z)=a^r g(N(\overline{x,\bar{y}}),z)=-a^r g(N(\bar{x},\bar{y}),z)=a^r N(x,y,z)$$

The remaining result can be proved similarly.

Corollary 1: If we put $P(x,y,z)\overset{def}{=}g(P(x,y),z)\, and\, T(x,y,z)\overset{def}{=}g(T(x,y),z)\, then$

$$N(x,y,z)+a^r P(x,y,z)+P(\bar{x},\bar{y},z)=0 \text{ and}$$

$$N(x,y,z)+a^r T(x,y,z)+T(\bar{x},\bar{y},z)=0$$

Consequently, if the given Hermite structure is subordinate to an almost tangent structure then

$$P(\bar{x},\bar{y},z)=T(\bar{x},\bar{y},z)=0$$

Corollary 2 :

$$T(\bar{x},y,z)=T(x,\bar{y},z) \tag{2.6a}$$

$$T(\bar{x},\bar{y},z)=a^r T(x,y,z) \tag{2.6b}$$

$$T(\bar{x},\bar{y},\bar{z})=-a^r T(x,y,\bar{z}) \tag{2.6c}$$

The relation (2.5c) implies that $N(x,y,z)$ is pure in all the slots and it follows from (2.6b) and (2.6c) that $T(x,y,z)$ is pure in x, y and $T(x,y,\bar{z})$ is hybrid x,y.

Theorem 2.4

Let us put

$$M(x,y)\overset{def}{=}D_x\bar{y}+a^r D_x y-\overline{D_x\bar{y}}-\overline{D_{\bar{x}}y} \tag{2.7a}$$

$$M(x,y,z)\overset{def}{=}-a^r g(M(x,y),z)=-g(M(\bar{x},\bar{y}),z) \tag{2.7b}$$

then

$$N(x,y)=M(x,y)-M(y,x) \tag{2.8a}$$

$$N(x,y,z)=M(x,y,z)-M(y,x,z) \tag{2.8b}$$

$$M(x,y,z)=-a^r\left\{-(D_{\bar{x}}f)(y,z)+(D_x f)(\bar{y},z)\right\} \tag{2.8c}$$

consequently, $M(x,y,z)$ is skew-symmetric in y, z:

$$M(x,y,z) = -M(x,z,y) \tag{2.8d}$$

Proof: From (2.7 a) we have

$$M(x,y) - M(y,x) = D_{\bar{x}}\bar{y} + a^r\overline{D_x y} - D_{\bar{x}}y - \overline{D_{\bar{x}}y} - D_{\bar{y}}\bar{x} - a^r D_y x + D_y\bar{x} + \overline{D_{\bar{y}}x}$$

$$= (D_{\bar{x}}\bar{y} - D_{\bar{y}}\bar{x}) + a^r(D_x y - D_y x) - (D_x\bar{y} - D_y\bar{x}) - \overline{(D_{\bar{x}}y - D_{\bar{y}}x)}$$

We know that $D_x y - D_y x = [x,y] + S(x,y)$

Using this result, we get (2.8a)

$$M(x,y,z) - M(y,x,z) = -a^r y(M(x,y) - M(y,x),z) = a^r g(N(x,y),z) = N(x,y,z)$$

which is nothing but (2.8)b) we have $(D_x F)(y) = D_x\bar{y} - \overline{D_x y}$. Using this result in (2.7a, b), we get

$$M(x,y) = (D_{\bar{x}}F)(y) - \overline{(D_x F)(y)} \tag{2.9a}$$

$$M(x,y,z) = -a^r g((D_x F)(y),z) + a^r f((D_x F)(y),z) \tag{2.9b}$$

It is easy to show that

$$g((D_x F)(y),z) = (D_{\bar{x}} f)(y,z) \tag{2.10a}$$

$$f((D_x F)(y),z) = -(D_x f)(\bar{y},z) = -(D_x f)(y,\bar{z}) \tag{2.10b}$$

Using these results in (2.9b), we get (2.8c). The relation (2.10b) obviously implies that $M(x,y,z)$ is skew-symmetric in y,z.

Corollary 1:

$$M(x,y,\bar{z}) = M(\bar{x},y,z) = M(x,\bar{y},z) \tag{2.11a}$$

$$M(x,\bar{y},\bar{z}) = M(\bar{x},y,\bar{z}) = M(\bar{x},\bar{y},z) = a^r M(x,y,z) \tag{2.11b}$$

Consequently, M is pure in all the slots and $M(x,y,\bar{z})$, $M(x,\bar{y},z)$ and $M(\bar{x},y,z)$ are all skew-symmetric in y,z.

Corollary 2 :

$$N(x,y,z) + N(y,z,x) + N(z,x,y) = 2M(x,y,z) + 2M(y,z,x) + 2M(z,x,y) \tag{2.11c}$$

$$M(x,y,z)+M(y,z,x)+(Mz,x,y) = -a^r\{(D_{\bar{x}}f)(y,z)+(D_{\bar{y}}f)(z,x)+(D_{\bar{x}}f)$$
$$(x,y)+(D_x f)(\bar{y},z)+(D_y f)(\bar{z},x) \tag{2.11d}$$
$$+(D_z f)(\bar{x},y)$$

Theorem 2.5

A necessary and sufficient condition that $M(x,y,z)$ be skew-symmetric in all the indices is

$$(D_{\bar{x}}F)(y)+(D_{\bar{y}},F)(x) = \overline{(D_x F)(y)} + \overline{(D_y F)(x)} \tag{2.12}$$

Proof : From (2.7), we have

$$M(x,y,z) + M(y,x,z,) = 0$$

$$D_{\bar{x}}\bar{y} + a^r D_x y - \overline{D_x \bar{y}} - \overline{D_{\bar{x}} y} + D_{\bar{y}}\bar{x} + a^r D_y x - \overline{D_y \bar{x}} - \overline{D_{\bar{y}} x} = 0 \tag{2.13}$$

but

$$D_{\bar{x}}\bar{y} - \overline{D_x \bar{y}} - \overline{D_y \bar{x}} + D_{\bar{y}}\bar{x} = (D_{\bar{x}}F)(y) + \overline{D_{\bar{x}} y} - (\overline{D_x F(y)} - a^r D_x y - (\overline{D_y F})(x)$$
$$- a^r D_y x + (D_{\bar{y}}F)(x) + \overline{D_{\bar{y}} x} \tag{2.14}$$

Substituting from (2.14) in (2.13), we get (2.12). This fact together with (2.8b) proves the theorem.

Corollary : A necessary and sufficient condition that $N(x,y,z)$ be skew symmetric in all indices is (2.12).

Proof : The statement follwings from (2.11 c) and theorem 2.5.

REMARKS

(a) If for an H-structure $((D_{\bar{x}}F)(y)=0$ is satisfied, then we say that V_n is Kahler space. It follows immediately from (2.9a) and (2.8a) that for a Kahler space $M(x,y)$ and $N(x,y)$ both vanish. Consequently. $M(x,y,z)$ and $N(x,y,z)$ both vanish

(b) If for an H-structure $(D_x F)y + (D_y F)x = 0$ is satisfied, then we say that V_n is an almost Tachibana space in the board sense M and N are both completely skew-symmetric.

(c) If V_n is equipped with an almost Hermite structure subordinate to an almost tangent structure, then it follows from (2.5a) and (2.7b) that $N(x,y,z)$ both vanish.

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AN INTERESTING OBSERVATION ON THE SUMS OF POWERS OF NATURAL NUMBERS THROUGH GENERATING FUNCTIONS

Umesh Chandra Sharma* & Bothania. F. Bokatow*

(Received 29.11.2005, Revised 08.08.2006)

ABSTRACT

Formulae for finding the sum of the k^{th} power of the first n natural numbers may find their use in some applications. The formulae for $k = 1,2,3$ are commonly found in the literature. For higher values of k, the formulae in terms of *B*ernoulli numbers and Stirling numbers of the second kind are available in the literature [7]. Srivastava et al. [8] have given an interesting account of applications of these formulae in Fractional Calcus. Venkatachalam & Sharma [3,4,5,6] have discussed various simple independent methods to derive the formulae. These formulae are in the form of polynomials. Let $s_k(n)$ represent the polynomial that represents $\sum_{i=1}^{n} i^k$. It is observed that $n(n+1)(2n+1)$ is a factor of $s_k(n)$ whenever k is an even positive integer and $n^2(n+1)^2$ is a factor of $s_k(n)$ whenever k is an odd positive integer ≥ 3. Here we prove this phenomenon by using the technique of generating functions and discuss two interesting methods to obtain the formulae in the factorized form.

INTRODUCTION

The formulae for the sum $s_k(n) = \sum_{i=1}^{n} i^k$ are commonly found in the literature for k=1,2,3. Most recently Venkatachalam & Sharma [4] discovered some simple methods to obtain the formulae and discussed various applications of them in Statistics, Game Theory, and Mechanical Engineering (Determination of moment of inertia, Design of riveted joints, Optimization in engineering designs). Venkatachalam & Sharma [3] also discovered some other interesting and simple methods to obtain the formulae. Giorgio Goldoni [2] has discussed very interesting and eye-catching results.

Let $F(x,t)$ be a function, which has a formal power series expansion in t i.e.

$$F(x,t) = \sum_{n=10}^{\infty} f_n(x)t^n$$

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The coefficient of t^n is a function of x and n. If this happens then we say that the expansion of $F(x,t)$ has generated the sequence $f_n(x)$. Victor Bryant [1] has explained the use of generating fuctions to find various formulae for Fibonacci numbers. The authors have tried the same technique on $s_k(n)$.

GENERATING FUNCTION FOR n^i

We know that

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n \text{ for } -1<x<1$$

Differentiating both sides with respect to x we get

$$\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1}$$

Multiplying both sides with x we get

$$\frac{x}{(1-x)^2}=\sum_{n=1}^{\infty}nx^n$$

Again we differentiate both sides with respect to x and get

$$\frac{1+x}{(1-x)^3}=\sum_{n=1}^{\infty}n^2x^{n-1}$$

Multiplying both sides with x we get

$$\frac{x+x^2}{(1-x)^3}=\sum_{n=0}^{\infty}n^2x^n$$

This process can be repeated as many times as we wish. In general we can find a polynomial $p_i(x)$ of degree i, without constant term, such that

$$\frac{p_i(x)}{(1-x)^{i+1}}=\sum_{n=0}^{\infty}n^ix^n \qquad \text{(A)}$$

Let $\alpha_{i,j}$ represents the coefficient of x^j in $p_i(x)$. Since $p_1(x)=x$, we have

$$\alpha_{1,1} = 1, \alpha_{1,j} = 0 \text{ if } j \neq 1. \tag{B}$$

For higher values of *i*, we can use the formula

$$\alpha_{i,j} = j\alpha_{i-1,j} + (i+1-j)\alpha_{i-1,j-1} \quad (i \geq 2) \tag{C}$$

which is obtained by equating the coefficient of x^j on both sides of

$$p_i(x) = x(1-x)^{i+1} \frac{d}{dx}\left\{\frac{p_{i-1}(x)}{(1-x)^i}\right\}$$

Using (B) and (C), various values of $\alpha_{i,j}$ can be displayed by a Pascal like triangle. Let us call that triangle as Alpha-Triangle.

The following table gives the Alpha-Triangle up to 10 rows.

	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	4	1							
4	1	11	11	1						
5	1	26	66	26	1					
6	1	57	302	302	57	1				
7	1	120	1191	2416	1191	120	1			
8	1	247	4293	15619	15619	4293	247	1		
9	1	502	14608	88234	156190	88234	14608	502	1	
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

We see that the entries in the triange grow very rapidly, and the palindrome nature of each row is eye-catching. The first row contains the column numbers and the first column contains the row numbers. The palindrome nature of each row can be proved by Mathematical Induction as follows.

Θ $\alpha_{1,1} = 1$ and $\alpha_{1,j} = 0$ if $j \neq 1$, the palindrome nature is true for $i = 1$. Let the

palindrome nature of numbers at the $(k-1)^{th}$ row be true i.e., $\alpha_{k-i,j} = \alpha_{k-1,k-j}$ (Induction hypothesis). From (C) we have

$$\alpha_{k,k-j+1} = (k-j+1)\alpha_{k-1,k-j+1} + j\alpha_{k-1,k-j}$$

$$= (k-j+1)\alpha_{k-1,j-1} + j\alpha_{k-1,j} \text{ (by Induction hypothesis)}$$

$$= j\alpha_{k-1,j} + (k+1-j)\alpha_{k-1,j-1}$$

$$= \alpha_{k,j} \text{ (From (C))}$$

which shows the palindrome nature of numbers at k^{th} row. Hence

$$\alpha_{k,k-j+1} = \alpha_{k,j} \text{ for } k \geq 1 \text{ and } j=1,\ldots,k. \tag{D}$$

These have many interesting properties. We state and prove some of them.

Theorem 1.

If k is an odd positive integer ≥ 3 then

$$\sum_{j=1}^{k}(-1)^j(j-1)!(k+1-j)!\alpha_{k,j} = \sum_{j=1}^{k}(-1)^j(k-j)!\,j!\,\alpha_{k,j} = 0 \tag{E}$$

Proof:

Let k be odd and > 3.

Then $$\sum_{j=1}^{k}(-1)^j(j-1)!(k+1-j)!\alpha_{k,j}$$

$$= \sum_{j=k}^{1}(-1)^{k-j+1}(k-J+1-1)!\,J!\,\alpha_{k,k-J+1}$$ (by a substitution $j = k - J + 1$ so that $J=1$ when $j = k$ and $J = k$ when $j = 1$)

$$= (-1)^{k+1}\sum_{J=1}^{k}(-1)^J(k-J)!\,J!\,\alpha_{k,J} \qquad \text{(From (D))}$$

$$= \sum_{j=1}^{k} (-1)^j (k-j)!\, j!\, \alpha_{k,j} \quad (j \text{ being dummy variable and } k \text{ being odd integer})$$

Also $\sum_{j=1}^{k} (-1)^j \; (k-j)!\, j!\, \alpha_{k,j}$

$$= \sum_{j=1}^{k} (-1)^j (k-j)!\, j!(j\alpha_{k-1,j} + (k+1-j)\alpha_{k-1,j-1}) \qquad \text{(From (C))}$$

$$= \sum_{j=1}^{k} (-1)^j (k-j)!\, j!\, j\alpha_{k-1,j} + \sum_{j=1}^{k} (-1)^j (k-j)!\, j!(k+1-j)\alpha_{k-1,j-1}$$

$$= \sum_{j=1}^{k-1} (-1)^j (k-j)!\, j!\, j\alpha_{k-1,j} + \sum_{j=2}^{k} (-1)^j (k-j)!\, j!(k+1-j)\alpha_{k-1,j-1}$$

$$\sum_{j=1}^{k-1} (-1)^j (k-j)!\, j!\, j\alpha_{k-1,j} + \sum_{j=1}^{k-1} (-1)^{j+k} (k-j-1)!(j+1)!(k-j)\alpha_{k-1,j}$$

(by a substitution $j - 1 \to j$)

$$= \sum_{j=1}^{k-1} (-1)^j \{(k-j)!\, j!\, j - (k-j-1)!(j+1)!(k-j)\}\alpha_{k-1,j}$$

$$= \sum_{j=1}^{k-1} (-1)^j [(k-j)!\, j!\{j-(j+1)\}]\alpha_{k-1,j}$$

$$= -\sum_{j=1}^{k-1} (-1)^j [(k-j)!\, j!]\alpha_{k-1,j} \quad = -S$$

$$= -\sum_{j=1}^{k-1} (-1)^j [(k-j)!\, j!]\alpha_{k-1,k-1-j+1} \qquad \text{(From (D))}$$

$$= -\sum_{j=1}^{k-1} (-1)^j (k-j)!\, j!\, \alpha_{k-1,k-j}$$

$$= -\sum_{j=1}^{k-1}(-1)^{k-j}(k-j)!\,j!\,\alpha_{k-1,j} \text{ (by substitution } k-j \to j)$$

$$= \sum_{j=1}^{k-1}(-1)^{j}(k-j)!\,j!\,\alpha_{k-1,j} = S \qquad (\Theta(-1)^{k} = -1)$$

which implies that $-S = S$ i.e., $S = 0$

$$\therefore \quad \sum_{j=1}^{k-1}(-1)^{j}(k-j)!\,j!\,\alpha_{k-1,j} = 0. \text{ Hence proved.}$$

Another interesting property of α 's is stated and proved in the next theorem.

Theorem 2.

If k is an even positive integer, then

$$\sum_{j=1}^{k}(-1)^{j}(3\times5\times\ldots\times(2j-1))(3\times5\times\ldots\times(2k-2j+1))\alpha_{k,j} = 0 \qquad \text{(F)}$$

Proof :

Let k be an even positive integer. Then

$$\sum_{j=1}^{k}(-1)^{j}(3\times5\times\ldots\times(2j-1))(3\times5\times\ldots\times(2k-2j+1))\alpha_{k,j}$$

$$= \sum_{j=1}^{k}(-1)^{j}(3\times5\times\ldots\times(2j-1))(3\times5\times\ldots\times(2k-2j+1))\alpha_{k,k-j+1} \qquad \text{(From(D))}$$

Making suitable change in the dummy variable, the above is equal to

$$\sum_{k=1}^{k}(-1)^{k-j+1}(3\times5\times\ldots\times(2k-2j+1))(3\times5\times\ldots\times(2j-1))\alpha_{k,j}$$

$$= -\sum_{j=1}^{k}(-1)^{j}(3\times5\times\ldots\times(2k-2j+1))(3\times5\times\ldots\times(2j-1))\alpha_{k,j}\,(\Theta(-1)^{k-1} = --1)$$

This implies that $\sum_{j=1}^{k}(-1)^j(3\times5\times\ldots\times(2j-1))(3\times5\times\ldots\times(2k-2j+1))\alpha_{k.j}=0.$

This completes the proof.

GENERATING FUNCTION FOR $s_k(n)$.

Let k be a positive integer. Let $S_k(n)$ represent the polynomial that represent the

sum $\sum_{i=1}^{n} i^k$ The following results are well known.

$$s_1(n)=\frac{1}{2}n(n+1)$$

$$s_2(n)=\frac{1}{6}n(n+1)(2n+1)$$

$$s_3(n)=\frac{1}{4}n^2(n+1)^2$$

Venkatachalam & Sharma [5,6] discovered various methods to obtain the formulae for $s_k(n)$ for higher values of k. They also proved that $s_k(n)$ is a polynomial of degree $k+1$ and $n(n+1)$ is always a factor of $s_k(n)$. It is observed that $n(n+1)(2n+1)$ is a factor of $sk(n)$ whenever k is even and $n^2(n+1)^2$ is a factor of $s_k(n)$ whenever k is odd and ≥ 3. We are going to prove the same in this section. We shall use the technique of generating functions just like Victor Bryant [1] who used this technique to find various formulae for Fibonacci numbers and some other interesting results. Let $f(x,k)$ be a generating function for $s_k(n)$, then

$$f(x,k)=\sum_{n=10}^{\infty}s_k(n)x^n$$

$$=s_k(0)+\sum_{n=1}^{\infty}\{s_k(n-1)+n^k\}x^n$$

$$=s_k(0)+\sum_{n=1}^{\infty}s_k(n-1)x^n+\frac{p_k(x)}{(1-x)^{k+1}} \quad \text{(From(A))}$$

$$= xf(x,k) + \frac{p_k(x)}{(1-x)^{k+1}} (\Theta\ s_k(0) = 0).$$

Hence we get.

$$f(x,k) = \frac{p_k(x)}{(1-x)^{k+2}} \tag{G}$$

$\therefore\ s_k(n)$ is equal to the coefficient of x^n in the right hand side of (G) and hence we get

$$s_k(n) = \frac{1}{(k+1)!}\{\sum_{j=1}^{k} \alpha_{k,j} \prod_{r=1}^{k+1} (n-j+r)\} \tag{H}$$

From (H) we have

$$s_1(n) = \frac{1}{2} n(n+1)$$

$$s_2(n) = \frac{1}{6}\{n(n+1)(n+2) + (n-1)n(n+1)\}$$

$$= \frac{1}{6} n(n+1)(2n+1)$$

$$s_3(n) = \frac{1}{24}\{n(n+1)(n+2)(n+3) + 4(n-1)n(n+1)(n+2) + (n-2)(n-1)n(n+1)\}$$

$$= \frac{1}{4} n^2 (n+1)^2$$

ODD POWER: EVEN POWER

Since $s_k(n)$ is a polynomial of degree $k+1$ with $n(n+1)$ as a factor, we can write $s_k(n) = n(n+1)q_k(n)$, where $q_k(n)$ is a polynomial of degree $k-1$ (I)

A careful scrutiny reveals that the formula (H) can be written as

$$s_k(n) = \frac{1}{(k+1)!} n(n+1) \sum_{j=1}^{k} \alpha_{kj} \left\{ \prod_{r=1}^{j-1} (n-r) \right\} \left\{ \prod_{t=2}^{k+1-j} (n+t) \right\} \tag{J}$$

which gives

$$q_k(n) = \frac{1}{(k+1)!}\sum_{j=1}^{k}\alpha_{kj}\left\{\prod_{r=1}^{j-1}(n-r)\right\}\left\{\prod_{t=2}^{k+1-j}(n+t)\right\} \tag{K}$$

with the understanding that

$$\prod_{r=1}^{j-1}(n-r) = 1 \text{ whenever } j\text{-}1 < 1\text{, and}$$

$$\prod_{t=2}^{k+1-j}(n+t) = 1 \text{ whenever } k+1\text{-}j < 2\text{., and}$$

Theorem 3.

$n\,(n+1)$ is a factor of $q_k\,(n)$ whenever k is odd and ≥ 3. (L)

Proof:

Let k be odd and ≥ 3. It is easy to see that

$$q_k(0) = \frac{1}{(k+1)!}\sum_{j=1}^{k}\alpha_{k,j}(-1)^{j-1}(j-1)!(k+1-j)!$$

$$= \frac{-1}{(k+1)!}\sum_{j=1}^{k}\alpha_{k,j}(-1)^{j}(j-1)!(k+1-j)!$$

$$= 0 \;\; (\text{From}(E))$$

This proves that n is a factor of $q_k\,(n)$

Again it is easy to see that

$$q_k(-1) = \frac{1}{(k+1)!}\sum_{j=1}^{k}(-1)^{j-1}j!(k-j)!\alpha_{k,j}$$

$$= \frac{-1}{(k+1)!}\sum_{j=1}^{k}(-1)^{j}j!(k-j)!\alpha_{kj}$$

$$= 0 \;\; (\text{From}(E))$$

This proves that $(n+1)$ is a factor of $q_k(n)$. This completes the proof.

Theorem 4.

$(2n+1)$ is a factor of $q_k(n)$ whenever k is an even positive integer. (H)

Proof :

Let k be an even positive integer. From (K) it is easy to see that

$$q_k(-\tfrac{1}{2}) = \frac{1}{(k+1)!2^{k-1}} \sum_{j=1}^{k} (-1)^{j-1}(3\times5\times\ldots\times(2j-1))(3\times5\times\ldots\times(2k-2j+1))\alpha_{k,j}$$

$$= \frac{-1}{(k+1)!2^{k-1}} \sum_{j=1}^{k} (-1)^{j}(3\times5\times\ldots\times(2j-1))(3\times5\times\ldots\times(2k-2j+1))\alpha_{k,j}$$

$=0$ (From (F)). Hence proved.

CONCLUSION

$n^2(n+1)^2$ is a factor of $s_k(n)$ whenever k is odd and ≥ 3 and $n(n+1)(2n+1)$ is a factor of $s_k(n)$ whenever k is even.

Theorem 5.

$$s_{k+1}(n) = (n+1)s_k(n) - \sum_{i=1}^{n} s_k(i) \quad \text{for} \quad k \geq 1 \tag{N}$$

Proof:

$$\begin{aligned} s_{k+1}(n) = \sum_{i=1}^{n} i^{k+1} &= 1^{k+1} + 2^{k+1} + 3^{k+1} + \ldots + n^{k+1} \\ &= 1.1^k + 2.2^k + 3.3^k + \ldots + n.n^k \\ &= 1^k + 2^k + 3^k + \ldots + n^k \\ &\quad + 2^k + 3^k + \ldots + n^k \\ &\quad\quad + 3^k + \ldots + n^k \\ &\quad\quad\quad \ldots \\ &\quad\quad\quad + (n-1)^k + n^k \\ &\quad\quad\quad\quad + n^k \end{aligned}$$

$$= \sum_{i=1}^{n} \sum_{j=i}^{n} j^k$$

$$= \sum_{i=1}^{n} \{s_k(n) - s_k(i-1)\}$$

$$= \sum_{i=1}^{n} \{s_k(n) - s_k(i) + i^k\}$$

$$= ns_k(n) - \sum_{i=1}^{n} s_k(i) + \sum_{i=1}^{n} i^k$$

$$= (n+1)s_k(n) - \sum_{i=1}^{n} s_k(i). \quad \text{Hence proved.}$$

Since $n\ (n+1)$ is a factor of $s_k(n)$ we can write $s_k(n) = n\ (n+1)\ q_k\ (n)$ where $q_k\ (n)$ is a polynomial of degree k - 1. Let $b_{i,j}$ represent the coefficient of n^j in $q_k\ (n)$. Since $s_1\ (n) = n$ $(n+1)(\frac{1}{2})s_2(n) = n(n+1)(\frac{1}{6}+\frac{n}{3})$ etc, $b_{1,0} = \frac{1}{2} b_{2,0} = \frac{1}{6}, b_{2,1} = \frac{1}{3}$, etc. As $q_k\ (n)$ is a polynomial of degree k - 1, $s_k(n) = n(n+1)\sum_{j=0}^{k-1} b_{k,j} n^j, b_{k,j} = 0$ if $j < 0$ or $j \geq$ k. (O)

Theorem 6.

$$q_k(n) = \frac{(n+1)q_{k-1}(n) - \sum_{j=1}^{k-1}(b_{k-1,j-2} + b_{k-1,j-1})q_j(n)}{1 + b_{k-1,k-2}} \text{ for } k \geq 2 \qquad \text{(P)}$$

Proof :

Putting $s_k(i) = i(i+1)\sum_{j=0}^{k-1} b_{k,j} i^j$ in Eq. (N) we get

$$s_{k+1}(n) = (n+1)s_k(n) - \sum_{i=1}^{n}\left\{(i^2+i)\sum_{j=0}^{k-1} b_{k,j} i^j\right\}$$

$$= (n+1)s_k(n) - \sum_{i=1}^{n} \left\{ \sum_{j=0}^{k-1} b_{k,j} i^{j+2} + \sum_{j=0}^{k-1} b_{k,j} i^{j+1} \right\}$$

$$= (n+1)s_k(n) - \sum_{i=1}^{n} \left\{ \sum_{j=2}^{k+1} b_{k,j-2} i^{j} + \sum_{j=1}^{k} b_{k,j-1} i^{j} \right\}$$

$$= (n+1)s_k(n) - \sum_{i=1}^{n} \left\{ \sum_{j=1}^{k+1} (b_{k,j-2} + b_{k,j-1}) i^{j} \right\}$$

$$= (n+1)s_k(n) - \sum_{j=1}^{k+1} (b_{k,j-2} + b_{k,j-1}) s_j(n)$$

$$= (n+1)s_k(n) - \sum_{j=1}^{k} (b_{k,j-2} + b_{k,j-1}) s_j(n) - b_{k,k-1} s_{k+1}(n)$$

which gives

$(1+b_{k,k-1})s_{k+1}(n) = (n+1)s_k(n) - \sum_{j=1}^{k} (b_{k,j-2} + b_{k,j-1}) s_j(n)$ which further gives

$$(1+b_{k,k-1})n(n+1)q_{k+1}(n) = n(n+1)^2 q_k(n) - n(n+1)\sum_{j=1}^{k} (b_{k,j-2} + b_{k,j-1}) q_j(n)$$

i.e. $$(1+b_{k,k-1})q_{k+1}(n) = (n+1)q_k(n) - \sum_{j=1}^{k} (b_{k,j-2} + b_{k,j-1}) q_j(n)$$

i.e. $$q_{k+1}(n) = \frac{(n+1)q_k(n) - \sum_{j=1}^{k} (b_{k,j-2} + b_{k,j-1}) q_j(n)}{1+b_{k,k-1}}$$

Making suitable change in the dummy variable k ($(k \to k-1)$), this formula reduces to

$$q_k(n) = \frac{(n+1)q_{k-1}(n) - \sum_{j=1}^{k-1} (b_{k-1,j-2} + b_{k-1,j-1}) q_j(n)}{1+b_{k-1,k-2}}$$ for $k \geq 2$. Hence proved.

Theorem 7.

$$b_{k,r} = \frac{b_{k-1,r} + b_{k-1,r-1} - \sum_{j=r+1}^{k-1}(b_{k-1,j-2} + b_{k-1,j-1})b_{j,r}}{1 + b_{k-1,k-2}} \quad \text{for } k \geq 2 \tag{Q}$$

Proof :

The coefficient of n^r in $(n+1)q_{k-1}(n) = b_{k-1,r} + b_{k-1,r-1}$.

The coefficient of n^r in $\sum_{j=1}^{k-1}(b_{k-1,j-2} + b_{k-1,j-1})q_j(n) = \sum_{j=r+1}^{k-1}(b_{k-1,j-1} + b_{k-1,j-1})b_{j,r}$

$\therefore$ From (P) we get required formula. This completes the proof.

From (Q) and Mathematical Induction we can prove the following result

$$b_{k,k-1} = \frac{1}{k+1} \quad \text{for all } k = 1,2,3,\ldots\ldots \tag{R}$$

From (R) the formulae (P) and (Q) can be simplified as

$$q_k(n) = \frac{k}{k+1}\left\{(n+1)q_{k-1}(n) - \sum_{j=1}^{k-1}(b_{k-1,j-2} + b_{k-1,j-1})q_j(n)\right\} \text{for } k \geq 2. \tag{S}$$

And $$b_{k,r} = \frac{k}{k+1}\left\{b_{k-1,r} + b_{k-1,r-1} - \sum_{j=r+1}^{k-1}(b_{k-1,j-2} + b_{k-1,j-1})b_{j,r}\right\} \tag{T}$$

Using the fact that $b_{1,0}$ = ½ and (T) we can generate the values of b's as follows.

	0	1	2	3	4	5	6	7	8	9	10	11
1	$\frac{1}{2}$											
2	$\frac{1}{6}$	$\frac{1}{3}$										
3	0	$\frac{1}{4}$	$\frac{1}{4}$									
4	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{5}$								
5	0	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{6}$							
6	$\frac{1}{42}$	$-\frac{1}{42}$	$-\frac{1}{7}$	$\frac{1}{7}$	$\frac{5}{14}$	$\frac{1}{7}$						
7	0	$\frac{1}{12}$	$-\frac{1}{12}$	$-\frac{5}{24}$	$\frac{5}{24}$	$\frac{3}{8}$	$\frac{1}{8}$					
8	$-\frac{1}{30}$	$\frac{1}{30}$	$\frac{17}{90}$	$-\frac{17}{90}$	$-\frac{5}{18}$	$\frac{5}{18}$	$\frac{7}{18}$	$\frac{1}{9}$				

9	0	$-\frac{3}{20}$	$\frac{3}{20}$	$\frac{7}{20}$	$-\frac{7}{20}$	$-\frac{7}{20}$	$\frac{7}{20}$	$\frac{2}{5}$	$\frac{1}{10}$			
10	$\frac{5}{66}$	$-\frac{5}{66}$	$-\frac{14}{33}$	$\frac{14}{33}$	$\frac{19}{33}$	$-\frac{19}{33}$	$-\frac{14}{33}$	$\frac{14}{33}$	$\frac{9}{22}$	$\frac{1}{11}$		
11	0	$\frac{5}{12}$	$-\frac{5}{12}$	$-\frac{23}{24}$	$\frac{23}{24}$	$\frac{7}{8}$	$-\frac{7}{8}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{1}{12}$	
12	$-\frac{691}{2730}$	$\frac{691}{2730}$	$\frac{3859}{2730}$	$-\frac{3859}{2730}$	$-\frac{515}{273}$	$\frac{515}{273}$	$\frac{49}{39}$	$-\frac{49}{39}$	$-\frac{15}{26}$	$\frac{15}{26}$	$\frac{11}{26}$	$\frac{1}{13}$

The first row contains the column numbers and the first column contains the row numbers. The above triangle reveals very interesting phenomena as follows :

(i) The sum of the last two non-zero terms of any row is equal to ½ and the sum of the remaining terms is equal to 0 canceling themselves in pair wise fashion. the first row is an exemption which contains only one non-zero term which is equal to ½.

(ii) $\sum_{j=0}^{k-1}(-1)^j b_{k,j} = 0$ whenever k is an odd integer ≥ 3.

(iii) $\sum_{j=0}^{k-1}\frac{(-1)^j b_{k,j}}{2^j} = 0$ whenever k is an even positive integer.

From Theorem 3 and Theorem 4 we can prove (ii) and (iii). In order to prove (i) we first prove the following theorem.

Theorem 8.

$$b_{k,k-2} = \frac{k-1}{2(k+1)} \text{ for } k \geq 2.$$

Proof :

We shall use the method of Mathematical Induction.

Let $\qquad b_{t,t-2} = \frac{t-1}{2(t+1)}$ for $t = 2,3,\ldots, k-1$.

From (T)

$$b_{k,k-2} = \frac{k}{k+1}\left\{b_{k-1,k-2} + b_{k-1,k-3} - \sum_{j=k-1}^{k-1}(b_{k-1,j-2} + b_{k-1,j-1})b_{j,k-2}\right\}$$

$$= \frac{k}{k+1}\left\{\frac{1}{k} + \frac{k-2}{2k} - (b_{k-1,k-3} + b_{k-1,k-2})b_{k-1,k-2}\right\}$$

$$= \frac{k}{k+1}\left\{\frac{1}{k} + \frac{k-2}{2k} - \left(\frac{k-2}{2k} + \frac{1}{k}\right)\frac{1}{k}\right\}$$

$$= \frac{k-1}{2(k+1)} \cdot \text{ Hence proved.}$$

From (R) and (U) we get an interesting result

$$b_{k,k-1} + b_{k,k-2} = \frac{1}{2} \text{ for } k \geq 2. \tag{V}$$

Theorem 9.

If $k \geq 4$ and $k - r$ is an odd integer ≥ 3, then $b_{k,r} + b_{k,r-1} = 0$. (W)

Proof

Let $k \geq 5$ and $b_{t,r} + b_{t,r-1} = 0$ whenever $t - r$ is odd and ≥ 3 for $t = 1,2, ..., k$-1. Now let $k - r$ be odd and ≥ 3. Since $b_{j,r} = 0$ for $j \leq r$ from (T) we have

$$b_{k,r} = \frac{k}{k+1}\left\{b_{k-1,r} + b_{k-1,r-1} - \sum_{j=1}^{k-1}(b_{k-1,j-2} + b_{k-1,j-1})b_{j,r}\right\} \textit{ and}$$

$$b_{k,r-1} = \frac{k}{k+1}\left\{b_{k-1,r-1} + b_{k-1,r-2} - \sum_{j=1}^{k-1}(b_{k-1,j-2} + b_{k-1,j-1})b_{j,r-1}\right\}$$

$$\therefore b_{k,r} + b_{k,r-1} = \frac{k}{k+1}\left\{b_{k-1,r} + b_{k-1,r-1} - \sum_{j=1}^{k-1}(b_{k-1,j-2} + b_{k-1,j-1})(b_{j,r} + b_{j,r-1})\right\} \tag{X}$$

Now from the induction hypothesis it is easy to see that all terms under the summation in (X) are zero except for $j = r + 1$ and $j = k - 1$. $\therefore$ From (X), we get

$$b_{k,r} + b_{k,r-1} = \frac{k}{k+1}\left\{\begin{array}{c} b_{k-1,r} + b_{k-1,r-1} - (b_{k-1,r-1} + b_{k-1,r})(b_{r+1,r} + b_{r+1,r-1}) \\ -(b_{k-1,k-3} + b_{k-1,k-2})(b_{k-1,r} + b_{k-1,r-1}) \end{array}\right\}$$

Now using (V) we have

$$b_{k,r} + b_{k,r-1} = \frac{k}{k+1}\left\{b_{k-1,r} + b_{k-1,r-1} - \frac{1}{2}(b_{k-1,r-1} + b_{k-1,r}) - \frac{1}{2}(b_{k-1,r} + b_{k-1,r-1})\right\} = 0.$$

Hence proved.

Theorem 10.

$$\sum_{j=0}^{k-3} b_{k,j} = 0 \text{ whenever } k \geq 3.$$

Proof :

Let $k > 3$. Then

$$s_k(n) = n(n+1)\sum_{j=0}^{k-1} b_{k,j} n^j .$$

Since $s_k(1) = 1$, we have

$1 = 2\sum_{j=0}^{k-1} b_{k,j}$. which gives

$$\sum_{j=0}^{k-1} b_{k,j} = \frac{1}{2}.$$

we have already proved that

$b_{k,k-2} + b_{k,k-1} = \dfrac{1}{2}$. Hence

$\sum_{j=0}^{k-3} b_{k,j} = 0$. This completes the proof.

AN INDEPENDENT APPROACH

So far we have developed a method of obtaining q_k using $q_1, q_2, \ldots, q_{k-1}$. Now we are going to develop a method to obtain q_k independently. It is clear that

$s_k(n)$-$s_k(n$-$1) = n^k$.

The left hand side and right hand side of (Y) are polynomials in n. The right hand side contains only one term i.e. n^k. On the right hand side the coefficient of n^k is 1 and the coefficients of

n_i,i=1,2,...,k-1 each is 0. Putting $s_k(n) = n(n+1)\sum_{i=0}^{k-1} b_{k,i} n^i$ and

$s_k(n-1) = (n-1)n\sum_{i=0}^{k-1} b_{k,i}(n-1)^i$ on the left hand side of (Y) and collecting the coefficients of various powers of n, we get

$$\sum_{r=0}^{k-1}\left\{(r+2)b_{k,r} + (-1)^r \sum_{i=r+1}^{k-1}(-1)^1\binom{i+1}{r} b_{k,i}\right\} n^r = n^{k-1}$$

Equating the coefficients of equal powers of n on both sides we get

$$b_{k,r} = \frac{(-1)^{r+1}}{r+2}\sum_{i=r+1}^{k-1}(-1)^i\binom{i+1}{r}b_{k,j} \text{for } r = 0,1,2,\ldots,k-2 \text{ and } k \geq 2 \qquad \text{(Z)}$$

Using $b_{k,k-1} = \frac{1}{k+1}$ (Z), we get $b_{k,k-2} = \frac{k-1}{2(k+1)}$ for $k \geq 2$. Similarly we get $b_{k,k-3} = \frac{(k-3)(k-2)}{12(k+1)}$ for $k \geq 3$. We can discover as many formulae as we wish.

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STUDY OF CARYOPHYLLIDEAN (CAPINGENTIDAE: *PSEUDOBATRACHUS*) TAPEWORMS OF FRESH WATER FISHES OF BUNDELKHAND REGION OF MADHYA PRADESH, INDIA: PART-I

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ABSTRACT

The three specimens of caryophyllideans were collected from the intestines of siluroid fresh water fish, *Clarias batrachus* (Linn.) collected from Bsanala near Laundi (Lovepuri) district Chhatarpur (M.P.) India. After the laboratory examinations, we reach on the conclusion that it fall under genus, Pseudobatrachus [4] and identified as a new member of the family Capingentidae [2] provisionally accommodated as *Pseudobatrachus moolchandrai* sp. nov.

KEY WORD: Caryophyllidea; Bsanala; Clarias; Madhya Pradesh.

INTRODUCTION

During the study of piscian cestodes we come across this very important historical city Chhatarpur of Bundelkhand region of Madhya Pradesh that is famous for its stone carving at Khajuraho. On 2nd February 2007our team reached Laundi (Lovepuri) and collected various types of fresh water fishes with the help of fisherman. Two fresh water cat fishes, *Clarias batrachus* (Linn.) were found infected with three alike cestodes in their intestines. Morphological studies of worm revealed them to belong to a new species, Pseudobatrachus moolchandrai sp. nov. in the genus, Pseudobatrachus [4] family Capingentidae [2]; order Caryophyllidea [1].

MATERIAL AND METHOD

The alimentary canal of the host were removed and cut open in normal saline water. The parasites remain attached in the mucosa by the scolex. Parasites were scraped with sharp scalpel. Worms were stretched in luke warm water with the help of fine brush and later fix in 5% formalin. Whole mounts were stained in Mayres Haemalum and cleared in xylol. Only camera lucida drawings were made. All the measurements in millimeters otherwise stated.

DESCRIPTION

***Pseudobatrachus moolchandrai* sp. nov. (Figs. A-D)**

Medium sized, unsegmented worms, measure 11.7 - 18.7 (15.2) in length and 0.56 - 0.87 (0.72) in width. Scolex well developed, differentiated from neck. Scolex spoon shaped, unarmed, measures 0.63 – 1.06 X 0.44 - 0.70 (0.84 X 0.57). Very long neck measures 3.96 - 6.89 (5.42) in length and 0.14 - 0.18 (0.16) in width. Testes ovoid, numerous, measure 0.15 - 0.21 X 0.14 - 0.22 (0.18 X 0.18), scattered in medullary

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parenchyma, reaches up to the level of cirrus pouch. Cirrus pouch oval to round, median, measure 0.32 - 0.36 X 0.25 - 0.32 (0.34 X 0.22). Internal seminal vesicles bell shaped measure 0.11 - 0.15 X 0.13 - 0.19 (0.13 X 0.16). External seminal vesicle absent. Female genitalia posteriorly situated. Ovary 'H'-shaped, measure 0.69 - 1.19 X 0.19 - 0.76 (0.94 X 0.48), lateral lobes of ovary situated in cortex and medulla while isthmus in medullary region only. Isthmus located in anterior half of ovary. Receptaculum seminis absent. Vitelline follicles partly cortical and partly medullary, measure 0.0625 - 0.1787 X 0.0625 -0.0875 (0.12 X 0.075), extend below the level of cirrus pouch but never touches the ovarian lobes. Post-ovarian vitellaria absent. Uterus broad, coiled, non-glandular, measure 1.12 - 1.85 X 0.025 - 0.41 (1.48 X 0.22). Male and female gonopores separately situated at the base of cirrus pouch. Eggs oval, operculate, measure 0.033 - 0.054 X 0.021 - 0.039 (0.044 X 0.030). Excretory pore measures 0.21 - 0.26 X 0.05 - 0.062 (0.25 X 0.056).

DISCUSSION

Presently two species *Pseudobatrachus chandrai* [4] and *Pseudobatrachus madhyapradeshensis* [2], are included in the genus *Pseudobatrachus* [4].

The present form differs from *Pseudobatrachus chandrai* [4] in having simple scolex without grooves, narrower neck, larger number of smaller testes, bell shaped internal seminal vesicle, larger ovary with very long lateral lobes and vitelline follicles never touches the ovarian lobes. It differs from *Pseudobatrachus madhyapradeshensis* [2] in having larger worms, larger scolex without apical and accessory suckers, wider neck, larger testes, larger cirrus pouch, larger ovary with comparatively long lateral arm, absence of receptaculum seminis and operculate eggs (Table - 01).

Thus the present form differs from all the valid species of genus, Pseudobatrachus[4].

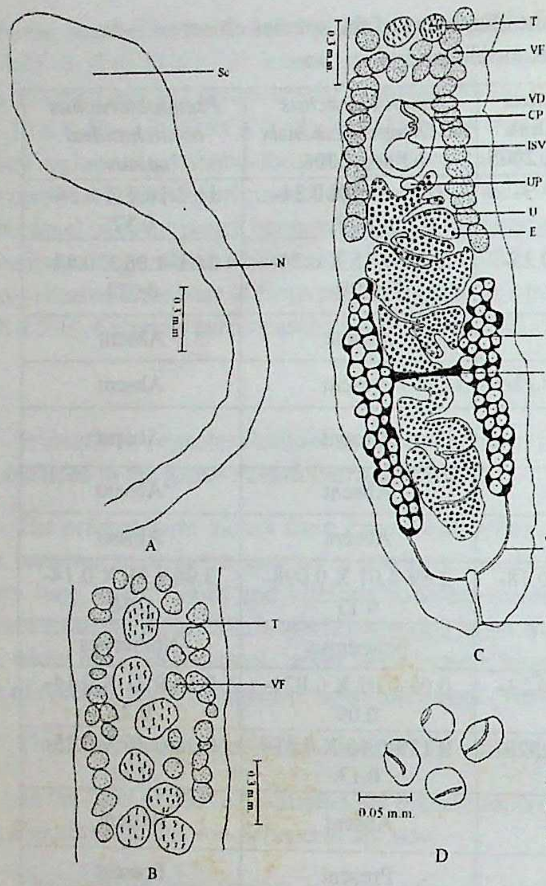
In the light of above discussion the present form may be provisionally accommodated as a new species, Pseudobatrachus moolchandrai sp. nov.

The name of new species is given after renound social worker, Mr. Mool Chandra Sahu of Bundelkhand Region of Madhya Pradesh.

Type species	:	*Pseudobatrachus moolchandrai* sp. nov.
Host	:	*Clarias batrachus* (Linn.)
Habitat	:	Intestine
Locality	:	Bsanala, Laundi (Lovepuri), District-Chhatarpur, (M.P.) India
Holotype	:	01
Paratype	:	02
Date of collection	:	28 February 2006
Accession number	:	BBCZD/HC/1054-1056
Deposition	:	Parasitological laboratory, Department of Zoology, Bipin Bihari (P.G.) College, Jhansi, (U.P.) India.

Table 01 :- Comparison of the characters of the species closer to Pseudobatrachus moolchandrai sp. nov.

S. No.	Characters		*Pseudobatrachus chandrai* Pathak and Srivastav, 2005	*Pseudobatrachus madhyapradeshensis* Khare, 2006	*Pseudobatrachus moolchandrai* sp.nov.
1.	Size of worms		6.0-20.0 X 0.9-1.056	10.0-16.0 X 0.34-0.50	11.7-18.7 X 0.56-0.87
2.	Scolex	Size	0.80-1.01 X 0.25-0.51	0.40-0.45 X 0.34-0.39	0.63–1.06 X 0.44-0.070
		Bothridea	Absent	Absent	Absent
		Apical Sucker	Absent	Present	Absent
		Accessory Sucker	Absent	Present	Absent
		Rostellum	Absent	Absent	Absent
		Grooves	Present	Absent	Absent
3.	Neck		4.51- 5.01 X 0.18-0.24	2.89-4.01 X 0.098-0.13	3.96-6.89 X 0.14-0.18
4.	Testes	Number	5-10	Numerous	Numerous
		Size	0.228-0.26X 0.23-0.328	0.05-0.07 X 0.074-0.09	0.15-0.21 X 0.14-0.22
5.	Cirrus pouch		0.314-0.4 X 0.328-0.4	0.139-0.18 X 0.11-0.13	0.32-0.36 X 0.25-0.32
6.	External seminal vesicle		Absent	Absent	Absent
7.	Internal seminal vesicle		Absent	Present	Present
8.	Ejaculatory duct		Present	Absent	Absent
9.	Ovary	Lateral lobes	Straight	Straight	Straight
		Size	0.642-0.8 X 0.6-0.8	0.58-0.68 X 0.23-0.30	0.69-1.19 X 0.19-0.76
10.	Vitellaria	Distribution	Touches to the ovarian lobes	Never touches to the ovarian lobes	Never touches to the ovarian lobes
		Size	0.070-0.12 X 0.084-0.128	0.026-0.039 X 0.026 -0.042	0.062-0.18 X 0.062-0.087
11.	Receptaculum seminis		Absent	Present	Absent
12.	Uterus		1.5-1.8 X 0.114-0.556	1.035-1.19X 0.18-0.244	1.12-1.85 X 0.025-0.41
13.	Eggs	Type	Operculate	Non-operculate	Operculate
		Size	0.025-0.041X0.05-0.0582	0.025-0.029X0.033-0.044	0.033-0.054X 0.021-0.039
14.	Host		*Clarias batrachus* (Linn.)	*Clarias batrachus* (Linn.)	*Clarias batrachus* (Linn.)
15.	Locality		Jalaun	Tikamgrah	Datia



Pseudobatrachus moolchandrai n. sp., A –Scolex (50X). B - Middle region of body (50X). C – Posterior region of the body (50X). D – Eggs (225X)

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ELECTROMAGNETIC FIELDS AND HUMAN HEALTH

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ABSTRACT

Domestic equipments including radio, TV and mobile phones and their transmission towers transmit low and high frequency electromagnetic field (EMF). The penetration of these fields into human body and their possible consequences are studied. Earlier studies have concluded that low (DC & ELF) and high frequency (UHF & Microwave) both types of EMF are harmful for cell or tissue life of human body. EMF is combination of both electric and magnetic fields. During the penetration of electromagnetic waves (EMW) inside the body, it is attenuated and due to this, voltage drop occurs across every type of cell or tissue, which may result into temperature rise in the body.

Key words : Electrical domestic equipments, Transmission towers, low and high frequency EMF, voltage drop, human body.

INTRODUCTION

Electromagnetic field (EMF) is a always present in the atmosphere. There are many sources of EMF; e.g. radio and TV transmission tower, air traffic control system, police and military radar system, satellite-broadcasting systems etc. All these sources transmit high frequency EMF. Whereas the domestic appliances transmit low frequency EMF e.g. stereo receiver, electric iron, refrigerator, toaster, hairdryer, color TV, coffee machine, vaccum cleaner, electric oven etc. Both high and low frequency EMF can affect the human health. The influences of both types of fields are differnt and are studied in this paper.

INFLUENCE OF LOW FREQUENCY ELECTROMAGNETIC FIELDS

Extensive work under the title is available on ifluence of extremely low frequency (ELF). Anselm et al. [6] claimed to observe improvement in drivig style by applying a DC field (1000 V at the ceiling electrode of the car) combined with a 10 Hz pulse field (20 V at ceiling electrode). Altman et al. [5] also observed an improvement in the concentration power of students under influence of DC field supported by square wave pulse field. But electric fields of little higher frequency range from 50 Hz to 100 Hz are of particular interest as these are the

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fields present in high intensities around the power generation equipments and transmission lines. People working in power houses and those living below and near transmission lines of high power are always exposed to these fields. Researches into the influence of these fields are of most concern to government and private power generation agencies, which try to influence the findings.

Some recent epidemiologic studies have suggested that exposure to electromagnetic fields (particularly ELF) increase the risk of leukemia and brain tumors [8,27,29,32,33]. In a statistical study on Finish population, Juutilainen et. al. [14] also found the increased incidence of these diseases in the people having jobs connected with EMF exposure.

The review of Sagan [26] gives a useful account of epidemiological and laboratory studies of these fields. It is seen that many studies of residential exposure have concluded of many fold increase of cancer risk among the children living around high power lines. Among the adults, the residential exposure is not found to increase the risk as much as the occupational exposure has been found to do so.

ELECTRIC APPLIANCES IN THE HOUSEHOLD

The strongest power frequency electric fields that are ordinarily encountered in the environment exist beneath high voltage transmission lines. In contrast, the strongest magnetic fields at power frequency are normally found very close to motors and other electrical appliances, as well as in specialized equipments such as magnetic resonance scanners used for medical imaging.

The domestic equipments transmit low frequency electric or magnetic or electriomagnetic field. The household ladies use maximum equipments within a distance of 30 cm . It means that they are exposed to low frequency electric and megnetic fields for considerably long time.

INFLUENCE OF HIGHER FREQUENCY FIELDS

For higher frequency fields, a lot of studies have been carried out with particular reference to harmful effects of radio and TV transmission towers. By a statistical survey, Rai[25] concluded that the people living around the radio and TV Transmission tower of Varanasi, suffered hazardous health problems. According to Feychting and Ahlbom [10,11] and Savitz et al. [28] high frequency EMF's affect the human body in many ways causing many diseases such as leukemia, brain tumor and cancer risk in childhood. These radiations affect the tissues, cells and skin of human body by induction of electric and magnetic fields inside the body on exposure to the radiation. It seems well suited to predict changes in thermoregulatory responses that may result from deposition of electromagnetic energy into the body and may

provid the basis for several such analysis [9,31,34]. Low intensity microwave fields influence thermoregulatory behaviour like other source of heat. Mice select a cooler part of a thermal gradient as the imposed RF field inside a wave-guide becomes more intense [13]. A discrete threshold power density governs such behavioral changes [2] and the behavior, once initiated, presists until the field is extinguished [3]. Michaelson and colleagues reported that dogs display a greater susceptibility to microwave heating after administration of pentobarbital sodium, morphine sulphate, or chlorpromazine, indicating that mechanisms of heat loss may be compromised by treatment with these drugs [21,24]. Efforts have been made to describe this type of studies by many workers like Kaune et al. [15], Mason [19], Gajsek et al. [11], Litwak et al. [18], Pathak et al. [23] and Kumar et al. [16,17].

MATERIAL AND METHODS

The mobile radio and TV transmission tower transmit radio frequency, high frequency and ultra high frequency electromagnetic fields in the atmosphere. The value of electric field around the transmission tower is given by (assuming antenna length d of few meters so that $d<<r$ and propagating wave front may be assumed to become spherical at large distance from antenna rod):

$$p / 4\pi r^2 = \varepsilon_0 E^2 c / 2$$

or $$E = \{P / 2\pi r^2 \varepsilon_0 c\}^{1/2} \quad (1)$$

where ε_0 is the permittivity of free space and c the speed of light (radiation). For 10 kW transmission tower, the electric field around tower is given by [23].

$$E = 774.6/r \quad (2)$$

When the high frequency electromagnetic field penetrates into the human body, the electric field inside the body decreases exponentially. The penetrated electric field inside the body at a distance z from surface is given by

$$E = E_0 \exp(-z / \delta) \quad (3)$$

where E_0 is the field the surface of the body and δ is the skin depth (i.e. depth at which the field is reduced to E_0/e) for the body. The variation of electric field inside the body is similar for skin and muscles but different for fat and bone.

When low or high frequency EMF propagates inside the body, a potential difference is produced across every tissue inside the body. Human body is a dielectric and so are the

tissues. Because of absorption of energy, the electric field gets reduced inside the body. It is applied across all the tissues and a potential difference created across them depends upon their shape and size. Tissues of linear dimension l in the field E will have potential drop

$$dU = El \tag{4}$$

The tissues are made up of cells. Because of potential drop, cell configuration may be disturbed. As the field is not static, it changes many times per second and the cells and tissues can feel the strees that changes the direction as many times per second.

Coming to the cell level, again it depends upon the size and shape of the cell. Only difference from the tissue is that there are no smaller constituents. Regarding the shape, some cells are nearly spherical, others are like tiny cubes, and still others are actually fussed so that they can scarcely be thought of as individual units. If we use a rectangular approximation to the cell, then the voltage drop across the cell is approximately given by Eq (4) [7].

Typically the length of rectangular cell varies from 10 μm to 150 μm. Its width used to be half the length and thickness is half the width. Similarly, spherical tissues of radius r will have potential drop [8]

$$d\,U = 1.5\, r\, E$$

MAGNETIC INDUCTION AND RESULTING STRESS

The magnetic field due to higher frequency EMW can be written as

$$B = E/c$$

Where, E is the electric field due to higher frequency EMF. The maximum value of E at a ditance of 10 m from the tower is 77.46 V/m. Thus the value of magnetic field strength due to EMF comes out to be negligible (of the order of 10^{-8} Vs/m^2. The domestic equipments also produce low frequency magnetic field. The value of magnetic field strength around all appliances repidly decreases as we get away from them. At a distance of 30 cm the magnetic fields surrounding most household appliances are more than 100 times lower than the given guideline limit of 100 μT at 50 Hz for the general public. Due to this the low as well as high frequency magnetic field is not expected to have any effect on human body.

RESULTS AND DISCUSSION

We know that an electric field and magnetic field are induced around all domestic equipments. When a biological body is exposed to EMF, electric field is penetrated inside the body and a potential drop is produced across every cell and tissue. If this electric field is parallel to the length of rectangular cell, the voltage drop occurs across the cells. The induced

eletric field is different if exposed electric field is parallel to width or thickness of rectangular cell. The voltage drop across cells or tissues vary because of variation of magnitude of incident field from various domestic equipments. The values of incident fields from various equipments has been taken from Federal Office for Radiation Safety (FORS) Germany (1999) publication who have given values at 30 cm distance from equipment operated by power at frequency of 50 Hz. The calculated values of voltage drop accross different sizes and shapes are given in Table 1.

Table 1

Voltage drop (in10^{-3}V) across the different shapes and size of the cell at 30 cm. from different domestic equipments.

Domestic Equipments / Cell Shape & Size	Stereo Receiver	Iron	Refrig-erator	Mixer	Toaster	Hair-dryer	Color TV	Coffee Machine	Vaccum Machine	Light
Rectangular, length 10 μ m	1.8	1.2	1.2	1.0	0.8	0.6	0.6	0.5	0.08	0.05
Rectangular, length 150 μm	27	18	18	15	12	9	9	7.5	1.2	0.75
Rectangular, length 75 μ m	13.5	9	9	7.5	6	4.5	4.5	3.75	0.6	0.37
Rectangular, length 5 μ m	0.9	0.6	0.6	0.5	0.4	0.3	0.3	0.25	0.04	0.02
Rectangular, thickness 2.5 μ m	0.45	0.3	0.3	0.25	0.2	0.15	0.15	0.125	0.02	0.01
Rectangular, thickness 37.5 μ m	6.75	4.5	4.5	3.75	3	2.25	2.25	1.875	0.3	0.18
Spherical, radius 5 μ m	1.35	0.9	0.9	0.75	0.6	0.45	0.45	0.375	0.06	0.03

Table 2

Voltage drop (in10^{-3}V) across the different shapes and size of the cell at different distances from the high frequency transmission tower

Cell Shape & Size \ Distance from the tower (m)	100	200	300	400	500	600	700	800	900	1000
Rectangular, length 10 µ m	.077	.0387	.0258	.0193	.0154	.0129	.011	.0096	.0086	.0077
Rectangular, length 150 µm	1.16	.58	.387	.29	.232	.1936	.165	.145	.129	.116
Rectangular, width 5 µ m	.038	.019	.0129	.0096	.0077	.0064	.005	.0048	.0043	.0038
Rectangular, width 75 µ m	.58	.29	.193	.145	.116	.0968	.082	.0726	.0645	.058
Rectangular, thickness 2.5 µ m	.019	.0096	.0064	.0048	.0038	.0032	.0027	.0024	.0121	.0019
Rectangular, thickness 37.5 µ m	.29	.145	.096	.0726	.058	0.484	.0414	.0363	.0322	.029
Spherical, diameter 5 µ m	.058	.029	.019	.014	.0116	.0097	.0083	.0072	.00645	.0058

Table 1 shows that the voltage drop across the cells is different due to different types of equipments. When any person comes near the domestic equipments, low frequency electric field penetrates inside the body and induces a potential difference across every tissue or cell. Every cell or tissue has a potential drop. It means that when electric field penetrates inside the body, its energy decreases continuously. The voltage drop is high for those equipments, which produce high electric field. The voltage drop across the rectangular cell of length 150 µ m is more due to the radiation of stereo receiver, electric iron and refrigerator. It means that energy absorption of this cell due to the radiation of these equpments is more. Voltage drop occurs across the cell due to the power absorption from EMF by every cell or tissue of the biological material. If the electric field of low frequency EMF is increased, the power absorption of cell or tissue also increases. Hence low frequency EMF is harmful for tissue and cell life of the body.

Similarly if any person comes near the high frequency transmission tower, he is exposed to the high frequency EMF. The induced electric field varies with distance from the tower. So the voltage drop across the cells or tissue also varies with distance from the tower. Table 2 illustrates that the voltage drop across the cells or tissues becomes high when a person comes near to the transmission tower. Thus the power absorption of cell is increased near the transmission tower. The power absorption of cell is higher for higher frequency of EMF. Due to the increment in power absorption of the cells or tissues, the temperature and conductivity of cell or tissue get increased when the thermoregulatory capabilitiy of the systems or parts of systems is exceeded. It may cause tissue damage and even death can result. As the absorbed energy steadily increases, the protective mechanisms for heart control breaks down, resulting in an uncontrolled rise in temperature. Michalson [20,21] and Michalson et al. [22] have demonstrated these effects in dogs and rats. Hence, high frequency EMF is harmful for tissues and cells of the body.

CONCLUSIONS

From the above analysis it is concluded that both high and low frequency EMF are harmful to the people living nearby the transmission tower or working for long near a domestic equipment. The harmfulness reduces as the distance increases from the tower or domestic equipments. Hence the electrical domestic equipments should remain as far as possible from human body and the transmission towers should be located away from the residential areas.

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SERUM CHARACTERISTICS OF THE FEATHERBACK TELEOST *NOTOPTERUS CHITALA* (Ham.)

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ABSTRACT

Normal serum levels of eight diagnostically important biochemical constituents in fish *Notopterus chitala* were observed, which depicts the state of internal environment of the fish. The values obtained for cholesterol, acid and alkaline phosphatase, total serum proteins, glutamic oxalacetic and pyruvic transaminase, cholinesterase and lactate dehydrogenase have been compared to some of those reported for other bony fishes. These indices may be helpful in assessing the hazardous health effects of aquatic contaminations in reference to biodiversity conservation.

INTRODUCTION

Blood parameters form an index to understand structural and functional status of the body organs in healthy as well as diseased fish [31] and play an important diagnostic tool in the assessment of physiopathological alterations under varying ecophysiological conditions. Physiological changes associated with adjustment to environmental conditions can be an indicator on physiological state of fish. Descriptive studies on the physicochemical properties of fish blood are few in comparison to mammals [12] and reports on less well investigated species are of interest for reference and serve to extend further observations [17]. Present paper deals with normal levels of eight biochemical blood constituents of the featherback teleost fish *Notopterus* itala, a commercially important food fish.

MATERIALS AND METHODS

Live fish were collected from river Gomti with the help of local fishermen and also brought from the fish markets of Lucknow and its suburbs. Fish were kept in large glass aquaria and allowed to rest for few hours. only healthy fishes were selected for these studies. Fish were taken out of aquarium wit least stress and blotted dry with a clean turkish towel. Blood was taken by severing off its caudal end and collected in a clean dry tube. Blood was allowed to clot and retract at room temperature for 30 minutes, then centrifuged at 2000 rpm and clear serum decanted in a vial and kept frozen until analyzed. Serum cholesterol level was estimated using method of Zlatkis *et al*, [35]. Serum acid and alkaline phosphatase, oxalacetic and pyruvic transaminase and total serum protein levels were determined following the method given by King and Wootton [15]. Serum cholinesterase and lactate dehydrogenase were analyzed as per methods of Hestrin [11] and King [14] given by Varley [33] respectively. Optical density was measured using Bausch and Lomb spectronic-20 colorimeter. Fresh blood film, body cavity and visceral organs of the fish were examined for any infection.

RESULTS AND DISCUSSION

Biochemical characteristics of blood are among the important indices of the state of internal

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environment of the fish [6]. Analysis of the serum constituents has proved useful in the detection and diagnosis of metabolic disturpances and diseases [1,18]. Results obtained on serum constituents of *N.chitala* have been summarized in Table-I. The report compares eight serum parameters to some of those reported for other bony fishes. High serum cholesterol levels reported in freshwater teleosts [5,9,10,13,23, 28-30] were apparen in *N. chitala.* The level obtained was close to those noticed in Cirrhina *mrigala*, *Labeo calbasu* and *L. gonius* [3]. Total serum protein levels are known to vary in different groups of fishes. The level noticed in *N. chitala* is comparable with those reported in *Wallago attu*, *Mystus seenghala*, *Rita rita*, *Channa marulius*, *Clarias batrachus* and *Heteropneustes fossilis* [27,28,32], but higher as compared to carps.

Phosphatases in general act upon a variety of phosphate esters [24] is associated with metabolism of carbohydrates and phosphoproteins [7], transportation and absorption [4,20]. Noda and Tachino[21] observed that alkaline phosphatase activity in various organs of brisk and bottom fishes were higher than in fresh water fishes. Serum alkaline and acid phosphatase levels noticed here in *N. chitala* are comparatively lower than active and hardy freshwater fishes, but almost in similar range with W. attu [31]. Such correlation is possibly due to their common food and feeding habits.

Blood parameters in fish are known to be influenced by various ecophysiological factors [8,19,25]. Enzymes like acetyl cholinesterase, lactate dehydrogenase and transaminases are helpful in determining the tolerance levels of xenobiotics in fishes, as they are the target enzymes involving blood, heart, liver, kidney and other tissues. Interaction of these enzymes due to environmental variations leads them to interplay between several substances and various metabolic pathways. Glutamic oxalacetic and glutamic pyruvic transaminases are known to actively take part in transamination reaction in body metabolism of living organisms, which allows an interplay between carpohydrate, fat and protein metabolism, to serve changing demands of the fish. Both glutamic oxalacetic and pyruvic transaminase levels observed in featherback were lower than noticed in freshwater major carps but within compafable range to catfishes such as *W. attu*, *M. seenghala* and *R. rita* [2]. Cholinesterase enzyme has great importance in neurophysiological activities of the body. It's evaluation in fishes is helpful in determining the tolerance levels of pesticides in its environment [22]. Lactate dehydrogenase being terminal enzyme in glycolysis path way catalyzing reversible transformation of lactate into pyruvate and plays an essential role in cellular metabolism. Both serum cholinesterase and lactate dehydrogenase activities in *N. chitala* were found in appreciable limits and can be well explained with the habit of the fish. Elevated lactate dehydrogenase levels are known to occur during anaemia, hepatitis and progressive muscular dystrophy. The results of present investigations are helpful in standardization of biochemical parameters to compute the health status of this economically important fish often used in aquaculture. Since the changes in biochemical blood profile is a mirror change in metabolism and physiological processes of the organism [16], these parameters can also be used in assessing hazardous effects of aquatic contamination leading to fast depleting population of fish in reference to biodiversity conservation.

Table-1: Biochemical Serum Constituents of *N.chitala*

Parameter		Mean + SD
1. Cholesterol	-	337.00 ± 86.24 mg/100 ml.
2. Alkaline phosphatase	-	03.42 ± 00.82 units/100 ml.
3. Acid phosphatase	-	14.76 ± 01.90 units/100 ml.
4. Total serum proteins	-	04.80 ± 01.04 gms/100ml.
5. Glutamic Oxalacetic Transaminase	-	34.60 ± 05.48 units/100ml.
6. Glutamic Pyruvic Transaminase	-	22.10 ± 02.70 units/100 ml.
7. Cholinesterase	-	51.03 ± 10.26 units/100ml
8. Lactate dehydrogenase	-	287:57 ± 38.90units/100ml.
No. of observations : 27, Weight range : 480 - 600 grams		

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ELECTRIC FIELDS INDUCED ON SURFACE OF AND INSIDE HUMAN BRAIN DUE TO THE MOBILE PHONE RADIATION

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ABSTRACT

The interaction of mobile phone radiation with human head is studies. It induces electric fields at the surface of and inside the head. For the determination of these, the calculations are made for electric fields (i) incident on human head surface, (ii) propagated through the scalp and skull to the brain surface and (iii) those penetrated into the brain. The variation of electric field with depth from brain surface is also evaluated.

Key Words: Electromagnetic fields (EMF), microwave radiation, skull, brain, human body, Federal communications commission (FCC) and Electromagnetic radiation (EMR).

INTRODUCTION

The worldwide expansion in the use of mobile telephones has lead to various studies of possible hazardous problems on human health. The human interaction with (EMF) from hand held set and transmission towers has been the center of attraction for many workers. The radiated power by the antenna of mobile phone is of order of 2 W [3]. Here we determine the penetration of radiated electric field from mobile phone, through the scalp and skull to the brain surface. The calculations are also made for propagation of electric field inside the brain.

THE ELECTRIC FIELLD ON THE SURFACE OF HUMAN HEAD

The side of human head is approximately plain; it has a layer of skin, fat and bones. The thickness of skull that covers the brain is in the range of 7-10 mm, measured with screw gauge by us in a laboratory (Shav Vichchhedan Bhawan) of State Ayurvadic College, Gurukula Kangri Haridwar (U.A.) with the help of Dr. Maher Singh Chauhan. The dielectric properties of the skull with scalp involve a relative permittivity and conductivity at mobile phone radiation. The value of electric field E_0 on the human head at a distance r from the mobile phone antenna of power P is given by (Polk [4]).

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$$P / 4\pi r^2 = E_0^2 \varepsilon_0 c / 2$$
$$E_0 = (P / 2\pi r^2 \varepsilon_0 c)^{1/2} \quad (1)$$
$$E_0 = 7.74\ P^{1/2} / r$$

where ε_0 is permittivity of free space and c is speed of light.

The power radiated from mobile phone antenna is the order of 2 W, so

$$E_0 = 10.954/r \quad (2)$$

Thus the electric field on the surface of head due to the antenna of cellular phones is inversely proportional to the distance from mobile phone antenna.

THE ELECTRIC FIEND INSIDE THE SKULL AND BRAIN

The electric field inside the skull due to the incident electric field 'E_0' on the surface of head at a distance z inside skull from the head is given by

$$E_z = E_0\ exp\ e^{-z\delta} \quad (3)$$

where δ is the skin depth (the distance at which the field is reduced to $1/e$ of its original value at the boundary). It depends upon the frequency of radiation for biological body and given by

$$\delta = 1 / q\omega$$
$$q = \left\{ \mu\varepsilon[(1+p^2)^{1/2}]/2 \right\}^{1/2} \quad (4)$$
$$p = \sigma / \omega\varepsilon$$

where ω is the radian frequency of radiation, ε the permittivity of tissue material. μ the permeability of tissue material σ the conductivity of tissue material. δ the skin depth. E_0 the electric field at the head surface.

The values of permittivity and conductivity are given by FCC at two frequencies 900 MHz and 1800 MHz. The value of electric fields and its absorption inside the skull is calculated for two frequencies with the dielectric properties $\varepsilon = 16.62\ \varepsilon_0$ & $\sigma = 0.2416$ S/m at 900 MHz and $\varepsilon = 15.56\ \varepsilon_0$ & $\sigma = 1.1530$ S/m at 1800 MHz, These values are taken from the Gabriel et al. [5]. After passing through the skull the electric field incident upon brain surface, this field for brain will be ε_0. Now using Eq. (1) we have calculated the variation of electric field inside the brain with dielectric properties $\varepsilon = 45.80\ \varepsilon_0$ & $\sigma = 0.7665$ S/m for 900 MHz and $\varepsilon = 43.54\ \varepsilon_0$ & $\sigma = 1.153$ S/m for 1800 MHz. Thus the electric field penetration and percentage absorption of field is calculated and given by Tables 1 & 2.

Table-1 : Variation of electric field with depth inside human brain at two frequencies.

Frequency of e.m. radiation (MHz)	Incident field on head E_0 (V/m)	Incident field on brain through skull and scalp E_1 (V/m)	Penetrated field inside brain (V/m) at different depth (cm)						
			1	2	3	4	5	6	7
900	547.7	507.02	410.76	332.78	269.60	218.42	176.95	143.46	116.14
1800	547.7	474.83	342.82	247.51	178.70	129.04	93.15	67.25	48.56

Table-2 : Variation of electric field absorption with depth inside human brain at two frequencies.

Frequency of e.m. radiation (MHz)	Incident field on head E_0 (V/m)	Incident field on brain through skull and scalp E_1 (V/m)	Percentae of absorbed electric field (V/m) at different depth (cm) Percentage of absorbed electric field at different depth (cm)						
			1	2	3	4	5	6	7
900	547.7	507.02	18.98	34.36	46.82	59.42	65.09	71.72	77.09
1800	547.7	474.83	27.80	47.87	62.36	72.82	80.38	85.33	89.77

Fig.1. Penetrated electric field with depth inside brain

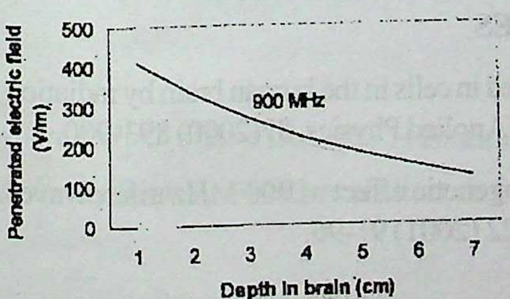
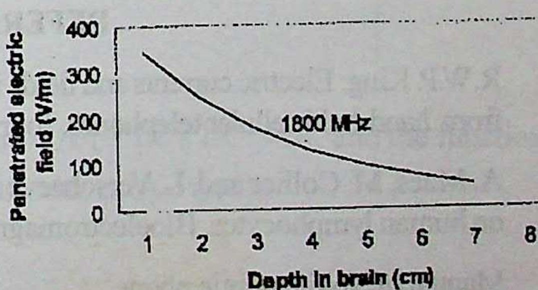


Fig.2. Penetrated electric field with depth inside brain



RESULT & DISCUSSIONS

The electric field induced in the brain by hand held cellular phones is detennined systematically. The steps are

I. The calculation of incident electric field on human head from antenna of cell phone.

II. The calculation of the electric field that penetrates through the scalp and skull and reaches the brain surface.

III. The calculation of the electric field that penetrates through the brain.

The variation of electric field with depth inside the brain is given in Fig. 1 & 2 for two generally used frequencies of mobile phones. These figures shows roughly linear decrease of the field inside the brain, which results in creation of voltage difference across the brain and tissues. The human skull and scalp absorbed the 7.5% of incident electric field from mobile phone antenna at frequency 900 MHz and double (approximately) absorption occurred at 1800 MHz. Thus the absorption of electric field is increases with increasing frequency.

To authors knowledge no experimental data for fields inside the brain due to mobile phone radiation exists in scientific literature. Theoretically, King [1] has calculated the variation of fields inside brain. He has taken the old models of mobile phone and output power to be 0.6 W for which he has obtained lower fields. However, the variation of fields is similar. Though, our formulation is very much simpler than his formuulation. It is difficult at this stage to predict the biological effects due to presence of these fields inside brain. Experimental observation by Maes [2] found no increased incidence of chromosome damage due to microwave exposure. But they found a significant increase in sister chromatid exchanges (SCE's) frequency due to microwave (900 MHz) exposure alongwith the chemical mutagen mitomycin C (MMC) exposure.

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EXISTENCE THEOREMS FOR A CLASS OF HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH ALGEBRAS

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ABSTRACT

In this paper, some results concerning the existence as well as existence of extremal solutions are proved for hyperbolic quadratic integro-differential equations using a nonlinear alternative of Leray-Schauder type and using a hybrid fixed point theorem on ordered Banach algebras due to Dhage [8, 9].

Key words and phrases: Hyperbolic differential equation, existence theorem.

AMS (MOS) Subject Classifications: 35L70, 35L20, 35R70.

INTRODUCTION

Let IR denote the real line. Given two closed and bounded intervals $J_a = [0,a]$ and $J_b = [0,b]$ in IR, consider the following hyperbolic integro-differential equation (in short HIDE)

$$\left.\begin{aligned} &\frac{\partial^2}{\partial x \partial y}\left[\frac{u(x,y)}{f(x,y,u(x,y))}\right] = g\left(x,y,u(x,y),\int_0^x\int_0^y k(x,y,t,s,u(t,s))dsdt\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad a.e.\,(x,y)\in J_a\times J_b \\ &u(x,0)=\varphi(x),\, u(0,y)=\psi(y) \end{aligned}\right\} \tag{1}$$

where $f: J_a \times J_b \times \mathrm{IR} \to \mathrm{IR} - \{0\}$ is continuous, $g: J_a \times J_b \times \mathrm{IR} \times \mathrm{IR} \to \mathrm{IR}$, and the functions $\varphi: J_a \to \mathrm{IR}, \psi: J_b \to \mathrm{IR}$ are continuous functions with $\varphi(0) = \psi(0)$.

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By a solution of the HIDE (1) we mean a function $u \in AC(J_a \times J_b, \mathrm{IR})$ satisfying

(i) the function $(x, y) \mapsto \left(\dfrac{u(x, y)}{f(x, y, u(x, y))} \right)$ is absolutely continuous, and

(ii) u satisfies the equations in (1),

where $AC(J_a \times J_b, \mathrm{IR})$ is the space of absolutely continuous real-valued functions on $J_a \times J_b$. The existence of solutions and the topological properties of the solutions set for the hyperbolic differential equations have received much attention during the last two decades ; we refer for instance to the papers by Dawidowski and Kubiaczyk [5, 6], De Blasi and Myjak [7], Kubiaczyk and Mostafa [13] and the references cited therein. Lakshmikantham and Pandit [15, 16] coupled the method of upper and lower solutions with the monotone method to obtain existence of extremal solutions for hyperbolic differential equations. The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems of the first and second order partial differential equations. We refer to the books by Carl and Heikkila [4], Heikkil¨a and Lakshmikantham [12], Ladde et al. [14], to the papers by Agarwal [1], Agarwal and Sheng [2], Blakley and Pandit [3], Lakshmikantham and Pandit [15], Pandit [16] and the references cited therein. The physical situations in which HIDE (1) occurs are yet to be investigated. However, HIDE (1) is new to the literature on the theory of hyperbolic partial differential equations and the study in this direction would be a good contribution to this area. This is a main motivation for the present paper. The rest of the paper is organized as follows. In the following section we present notations, definitions and preliminary results needed in the following sections. In Section 3, an existence theorem for solutions of the HIDE (1) is proved under certain Lipschitz and Carath´eodory conditions. Section 4 deals with the existence of extremal solutions. The results of the present paper extend in the Banach algebra setting those results considered in the previous literature for the particular problem (1) with $f \equiv 1$.

PRELIMINARIES

In this section, we introduce notations and definitions which will be used in the sequel. Let $B(J_a \times J_b, \mathrm{IR})$ denote the space of bounded functions on $J_a \times J_b$ and let $C(J_a \times J_b, \mathrm{IR})$ be the Banach space of all continuous functions from $J_a \times J_b$ into IR with the norm

$$\|u\|_\infty = \sup\{|u(x, y)| : (x, y) \in J_a \times J_b\}. \tag{2}$$

Define a multiplication "." by

$$(u.v)(x,y) = u(x,y)\, v(x,y)$$

for each $(x,y) \in J_a \times J_b$. Then $C(J_a \times J_b, \mathrm{IR})$ is a Banach algebra with above norm and

multiplication. Let L_1 ($J_a \times J_b$, IR) denotes the Banach space of measurable functions u : $J_a \times J_b \to$ IR which are Lebesgue integrable normed by

$$\|u\|_{L^1} = \int_0^a \int_0^b |u(x,y)| dy\, dx.$$

The HIDE (1) is equivalent to the functional integral equation (in short FIE).

$$u(x,y) = [f(x,y,u)](z_0(x,y) + \int_0^x \int_0^y g(t,s,u(t,s)), \int_0^t \int_0^s k(t,s,\tau,\eta,u(\tau,\eta))d\eta\, d\tau)\, ds\, dt) \quad (3)$$

for $(x,y) \in J_a \times J_b$, where the function $z : J_a \times J_b \to$ IR is defined by

$$z_0(x,y) = \frac{\psi(y)}{f(0,y,\psi(y))} + \frac{\varphi(x)}{f(x,0,\varphi(x))} - \frac{\varphi(0)}{f(0,0,\varphi(0))}.$$

From the continuity of the functions f, φ and ψ it follows that $z_0 \in C(J_a \times t_b, \text{IR})$.

We shall apply the following nonlinear alternative of Leray-Schauder type, recently proved in Dhage [8] for proving the main existence result of this paper.

Theorem 2.1

Let $B_r(0)$ and $\overline{B}_r(0)$ respectively denote on open and closed ball in a Banach algebra X centered at origin of radius r, for some real number $r > 0$. Let $A : X \to X$ and B: $\overline{B}_r(0) \to X$ be two operators such that:

(a) A is Lipschitz with the constant k;

(b) B is completely continuous;

(c) $k\,M < 1$, where $M = \|B(\overline{B}_r(0))\| = \sup\{\|Bx\| : x \in \overline{B}_r(0)\}$.

Then either

(i) the equation $Ax\,Bx = x$ has a solution in $\overline{B}_r(0)$; or

(ii) there exists an $u \in X$ with $\|u\| = r$ such that $\lambda A\left(\frac{u}{\lambda}\right)Bu = u$ for some $0 < \lambda < 1$.

We need the following definition in the sequel.

Definition 2.1

A function $\beta : J_a \times J_b$ IR×IRI $\to$ IR is called caratheodory if

(i) the function $(x,y) \to (x,y,w,z)$ is measurable for each $w,z \in$ IR,

(ii) the function $(w,z) \to \beta(x,y,w,z)$ is continnous for almost each $(x,y) \in J_a \times J_b$.

Further a Caratheodory function $\beta(x,y,w,z)$ is called L^1— Caratheodory if

(iii) for each real number $q > 0$, there exists a function $h_q L^1 \in (J_a \times J_b, \mathrm{IR})$ such that

$$|\beta(x, y, w, z)| \leq h_q(x, y) a.e. (x, y) \in J_a \times J_b$$

for all $w,z \in \mathrm{IR}$ with $|w| \leq q$ and $|z| \leq q$.

EXISTENCE OF SOLUTIONS

In this section, we prove the existence of solutions for the problem (1) under certain monotonicity conditions. The following hypotheses will be used in the sequel.

(A1) The function f is continuous and there exists a positive function $\ell \in B(J_a \times J_b, \mathrm{IR})$ with bound $\|\ell\|_\infty$ such that

$$|f(x, y, z) - f(x, y, \bar{z}| \leq \ell(x, y)|z - \bar{z}|$$

for all $(x,y) \in J_a \times J_b$, and for all $z, \bar{z} \in \mathrm{IR}$.

(B0) The function k is continuous on $J_a \times J_b \times J_a \times J_b \times \mathrm{IR}$ into IR and there exists a function $\alpha \in L^1(J_a \times J_b, \mathrm{IR})$ such that

for all $x, t \in J_a, y, s \in J_b$ and $u \in \mathrm{IR}$.

(B1) The function g is Caratheodory.

(B2) There exists a function $\gamma \in L^1(J_a \times J_b, \mathrm{IR})$ such that $\gamma(x, y) > 0$ $a.e. (x, y) \in J_a \times J_b$ and a continuous nondecreasing submultiplicative function $\Omega : \mathrm{IR}^+ \to \mathrm{IR}^+$ Satisfying

$$|g(x, y, w, z)| \leq \gamma(x, y)\Omega(|w| + |z|), a.e. (x, y) \in J_a \times J_b, \text{ for all } w,z \in \mathrm{IR}.$$

Note that if the hypotheses (B0) and (B2) hold, then we have the following estimate concerning the function g. For any $u \in C(J_a \times J_b, \mathrm{IR})$, one has

$$\left|g\left(t, s, u(t, s), \int_0^t \int_0^s k(t, s, \tau, \eta, u(\tau, \eta))\, d\eta\, d\tau\right)\right|$$

$$\leq \gamma(t,s)\Omega\left(|u(t,s)| + \int_0^t \int_0^s |k(t,s,\tau,\eta,u(\tau,\eta))|\, d\eta\, d\tau\right)$$

$$\leq \gamma(t,s)\Omega\left(|u(t,s)| + \int_0^t \int_0^s \alpha(\tau,\eta)|u(\tau,\eta)|\, d\eta\, d\tau\right)$$

$$\leq \gamma(t,s)\Omega\left(\|u\|_\infty + \|\alpha\|_{L^1}\|u\|_\infty\right)$$

$$\leq \gamma(t,s)\left(1+\|\alpha\|_{L^1}\right)\Omega\left(\|u\|_\infty\right)$$

for all and $\iota \in J_a$ and $s \in J_b$.

Theorem 3.1

Assume that hypotheses (A1), (B0)-(B2) hold. Suppose that there exists a real number $r > 0$ such that

$$r > \frac{F\left[\|z_0\|_\infty + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\Omega(r)\right]}{1-\|\ell\|_\infty\left[\|z_0\|_\infty + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\Omega(r)\right]} \tag{4}$$

and

$$\|\ell\|_\infty\left[\|z_0\|_\infty + \|\gamma\|_{L1}\left(1+\|\alpha\|_{L1}\right)\Omega(r)\right] < 1 \tag{5}$$

where, $F = \sup_{(x,y)\in J_a\times J_b} |f(x,y,0)|, \|z_0\|_\infty = \left\|\frac{\psi(y)}{f(0,y,\psi(y))}\right\|_\infty + \left\|\frac{\varphi(x)}{f(x,0,\varphi(x))}\right\|_\infty + \left|\frac{\varphi(0)}{f(0,0,\varphi(0))}\right|$

with $\left\|\frac{\varphi(x)}{f(x,0,\varphi(x))}\right\|_\infty = \sup_{x\in J_a}\left|\frac{\varphi(x)}{f(x,0,\varphi(x))}\right|, \left\|\frac{\varphi(y)}{f(0,y,\psi(y))}\right\|_\infty = \sup_{y\in J_b}\left|\frac{\psi(y)}{f(0,y,\psi(y))}\right|$. Then the problem (1) has a solutions on $J_a \times J_b$ with $\|u\|_\infty \leq r$.

Proof. Let $X = C(J_a \times J_b, \mathrm{IR})$ and consider the closed ball $\overline{B}_r(0)$ in X centered at origin and of radius r, where the real number $r > 0$ satisfies the inequalities in (4). Define two operators $A: X \to X$ and $B: \overline{B}_r(0) \to X$ by

$$Au(x,y) = f(x,y,u(x,y)),\ (x,y) \in J_a \times J_b, \tag{6}$$

and

$$Bu(x,y) = z_0(x,y) + \int_0^x \int_0^y g\left(t,s,u(t,s), \int_0^t \int_0^s k(t,s,\tau,\eta,u(\tau,\eta)d\eta\, d\tau)\right)ds\, dt, \qquad (7)$$

for all $(x,y) \in J_a \times J_b$.

Now solving HIDE (1) is equivalent to FIE (3), which is further equivalent to solving the operator equation

$$Au\,(x,y)\; Bu\,(x,y) = u\,(x,y), \qquad (x,y) \in J_a \times J_b \qquad (8)$$

We show that operators A and B satisfy all the assumptions of Theorem 2.1. First we shall show that A is a Lipschitz operator on X. Let $u_1, u_2 \in X$. Then by (A1),

$$|Au_1(x,y) - Au_2(x,y)| = |f(x,y,u_2(x,y)) - f(x,y,u_2(x,y))|$$

$$\leq l(x,y,)|u_1(x,y) - u_2(x,y)|$$

$$\leq \|l\|_\infty \|u_1 - u_2\|_\infty$$

Taking the maximum over (x,y), in the above inequality yields

$$\|Au_1 - Au_2\|_\infty \leq \|l\|_\infty \|u_1 - u_2\|_\infty,$$

and so A is a Lipschitz constant $\|\ell\|_\infty$

Next, hypothesis $(B1)$ together the Lebesgue dominated concergence theoren implies that operator B: $\overline{Br}$ (0) $\longrightarrow$ X is continuous. We shall show that B is compact. Let $\{u_n\}$ be a sequence in $\overline{Br}$ (0). Then $\|u_n\| \leq r$ for each n $\in$ IN. From. (B2) it follows that

$$\|Bu_n\|_\infty \leq \|z_0\|_\infty + \|\gamma\|_{L^1}\left(1 + \|\alpha\|_{L^1}\right)\Omega(r)$$

for all $n \in$ IN. As a result $\{Bu_n : n \in \text{IN}\}$ is a uniformly bounded set in X. Let $(x_1, y_1), (x_2 y_2)$ $J_a \times J_b$. Then

$$|Bu_n(x_1 y_1) - Bu_n(x_2 y_2)| \leq |z_0(x_1 y_1) - z_0(x_2 y_2)|$$

$$+\left| \int_{x1}^{x2} \int_{y1}^{y2} \left| g\left(t,s,u,(t,s), \int_0^t \int_k^s k(t,s,\tau,\eta,u(\tau,\eta)d\eta\, d\tau)\right)\right| ds\ dt\right|$$

$$\leq |z_0(x_1 y_1) - z_0(x_2, y_2,)|$$

$$\leq \left|\int_{x1}^{x2} \int_{y1}^{y2} \gamma(t,s)(1 + \|\alpha\|_{L^1})\Omega(r) ds dt\right|$$

$$\to 0, as (x_1 y_2) \to (x_2 y_2).$$

From this we conclude that $\{Bu_n : n \in \mathbb{N}\}$ is an equicontinuous set in X. Hence $B: \overline{B_r}(0) \to X$ is compact by Arzela - Ascoli theorem.

Notice that

$$M = \left\| B(\overline{B}r(0)) \right\|$$

$$\leq \|z_0\|_\infty + \sup_{(x,y,)\in J_a\times J_b} \int_0^x \int_0^y \left| g\left(t,s,u,(t,s,), \int_0^t \int_0^s k(t,s,\tau,\eta,u(\tau,\eta)d\eta\, d\tau)\right)\right| ds\, dt$$

$$\leq \|z_0\|_\infty + \|\gamma\|_{L^1}\left(1+\|\alpha\|_\infty\right)\Omega(r),$$

and so, by (5)

$$kM \leq \|z_0\|_\infty + \|\gamma\|_{L^1}\left(1+\|\alpha\|_\infty\right)\Omega(r) < 1.$$

Now we are ready to apply Theorem 2.1. Note that conclusion (ii) cannot hold. To see this , let there be an $u \in X$ with $\|u\| = r$ satisfying $\lambda A\left(\frac{u}{\lambda}\right) Bu = u$ for some $0 < \lambda < 1$. Then for any $(x,y) \in J_a \times J_b$,

$$|u(x,y,)| \leq \left[\lambda\left| f(x,y,\frac{u}{\lambda}(x,y)) - f(x,y,0)\right| + \lambda\left| f(x,y,0)\right|\right]$$

$$\times\left(|z_0(x,y)| + \int_0^x \int_0^y \lambda(t,s)(1+\|\alpha\|_{L^1})\Omega(\|u\|_\infty) ds\, dt\right)$$

$$\leq \left[l(x,y,)\|u\|\infty + F\right]\left(|z_0(x,y,)| + \int_0^x \int_0^y \lambda(t,s)(1+\|\alpha\|_{L^1})\Omega(\|u\|_\infty) ds\, dt\right)$$

$$\leq \left[\|l\|\infty\|u\|\infty + F\right]\left(\|z_0\|\infty + \int_0^x \int_0^y \lambda(t,s)(1+\|\alpha\|_{L^1})\Omega(\|u\|_\infty) ds\, dt\right)$$

$$\leq \left[\|l\|\infty\|u\|\infty + F\right]\left(\|z_0\|\infty + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\Omega(\|u\|_\infty)\right)$$

$$\leq \|l\|\infty\|u\|\infty(\|z_0\|\infty + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\Omega(\|u\|_\infty))$$

$$+F\|z_0\|_\infty + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\Omega(\|u\|_\infty))$$

for all $(x,y,) \in J_a \times J_b$. Taking supremum over x and y we obtain

$$\|u\|_\infty \leq \|l\|_\infty\|u\|_\infty\left(\|z_0\|_\infty + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\Omega(\|u\|_\infty)\right)$$

$$+F\left(\|z_0\|_\infty + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\Omega(\|u\|_\infty)\right) \tag{9}$$

Substituting $\|u\|\infty = r$ in the above inequality yields

$$r \leq \|l\|_{\infty} r\left(\|z_0\|_{\infty} + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\right)\Omega(r) + F\|z_0\|_{\infty} + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\Omega(r)$$

or $$r \leq \frac{F\left(\|z_0\|\infty + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\Omega(r)\right)}{1-\|l\|\infty\left(\|z_0\|\infty + \|\gamma\|_{L^1}(1+\|\alpha\|_{L^1})\Omega(r)\right)}$$

This is a contradiction to (4). As. a result, the functional HIDE (1) has a solution u on $J_a \times J_b$. with $\|u\| \leq r$ This completes the proof.

4. EXISTENCE OF EXTREMAL SOLUTIONS

A nonempty closed set K in a Banach algebra X is called a cone if

(i) $K + K \subseteq K$,

(ii) $\lambda K \subseteq K$ *for* $\lambda \in IR, \lambda \leq 0$;

(iii) $\{-K\} \cap \{K\} = \{0\}$, where 0 is the zero element of X.

A cone K is called positive if (iv) $KoK \subseteq K$, where "o" is multiplication composition in X. We introduce an order relation $\leq$ in K as follows. Let $x, y \in X$. Then $x \leq y$ if and only $y - x \in K$. is called normal if the norm $\|.\|$ is monotone increasing on K. It is knowm that if the cone K is normal in K is normal in X, then every order- bounded set in X is norm - bounded. The details of cones and their properties appear in Guo and Lakshmikantam (11). the condition.

$$u \leq \bar{u} \text{ if and only if } u(x,y) \text{ for all } (x,y) \in J_a \times J_b$$

defines a partial ordering in $C(J_a \times J_b, \mathrm{IR})$. If $\underline{u} \leq \bar{u}$ we put

$$[\underline{u}, \bar{u}] = \{u \in C(J_a \times J_b, IR) : \underline{u} \leq u \leq \bar{u}\},$$

and call the set $[\underline{u}, \bar{u}]$ to be an order interval in $C(J_a \times J_b, \mathrm{IR})$. We equip the space $C(J_a \times J_b, \mathrm{IR})$ with the order relation $\leq$ with the help of the cone K defined by

$$K = \{u \in C(J_a \times J_b, \mathrm{IR}) : u(x,y) \geq 0, \forall (x,y) \in J_a \times J_b\}.$$

It is well-known that the cone K is positive and normal in $C(J_a \times J_b, \mathrm{IR})$. As a result of positivity of the cone K in $C(J_a \times J_b, \mathrm{IR})$ we have:

Lemma 4.1 (*Dhage* [9]). Let u_1, u_2, v_1, v_2 K be such that $u_1 \leq v_1$ and $u_2 < v_2$. Then $u_1 u_2 \leq v_1 v_2$.

Definition 4.1

A function $\underline{u}(.,.) \in C(J_a \times J_b, \mathrm{IR})$ is said to be lower solution of the $IIIDE$ (1) if the function

$(x,y)\mapsto\left(\dfrac{\underline{u}(x,y)}{f(x,y,\underline{u}(x,y))}\right)$ is absolutely continuous, and

$$\frac{\partial^2}{\partial x\partial y}\left[\frac{\underline{u}(x,y)}{f(x,y,\underline{u}(x,y))}\right]\le g\left(x,y,\underline{u}(x,y),\int_0^x\int_0^y k(x,y,t,s,\underline{u}(t,s))ds\,dt\right),$$
$$a.e.(x,y)\in J_a\times J_b$$

and $\underline{u}(x,0)\le\varphi(x),\underline{u}(0,y)\le\psi(y)$

for all $(x,y)\in J_a\times J_b$. Similarly, a function $\overline{u}(.,.)\in C(J_a\times J_b,\mathrm{IR})$ is said to be an upper solution of the HIDE (1) if the function $(x,y)\mapsto\left(\dfrac{\overline{u}(x,y)}{f(x,y,\overline{u}(x,y))}\right)$ is absolutely continuous,

and $$\frac{\partial^2}{\partial x\partial y}\left[\frac{\overline{u}(x,y)}{f(x,y,\overline{u}(x,y))}\right]\ge g\left(x,y,\overline{u}(x,y),\int_0^x\int_0^y k(x,y,\overline{u}(t,s))dsdt\right),$$
$$a.e.(x,y)\in J_a\times J_b$$

and $\overline{u}(x,0)\ge\varphi(x),\overline{u}(0,y)\ge\psi(y)$

for all $(x,y)\in J_a\times J_b$.

Definition 4.2

A solution u_M of problem (1) is said to be maximal if for any other solution u to the problem (1) one has $u(x,y)\le u_M(x,y),\forall(x,y)\in J_a\times J_b$. Again a solution u_m of the problem (1) is said to be minimal if $u_M(x,y)\le u(x,y),\forall(x,y)\in J_a\times J_b$ where u is any solution of the problem (1) on $J_a\times J_b$.

Definition 4.3

Let X be a ordered Branch space. A operator $T:X\to X$ is said to be nondecreasing if for all $u_1,u_2\in X,u_1\le u_2,$ we have that $Tu_1\le Tu_2$.

Theorem 4.1 [9]

Let $[\underline{u},\overline{u}]$ be the order interval in a Banach algebra X with cone K. Suppose that A, B: $[\underline{u},\overline{u}]\to K$ are two operators such that

(a) A is Lipschitz with the Lipschitz constant k,

(b) B is completely continuous,

(c) $AuBu \in [\underline{u}, \overline{u}]$ for each $u \in [\underline{u}, \overline{u}]$, and

(d) A and B are nondecreasing.

Furthermore, if the cone K is positive and normal, then the operator equation $Au\,Bu = u$ has the least and the greatest positive solutions in $[\underline{u}, \overline{u}]$, whenever $kM<1$. where $M = \|B([\underline{u}, \overline{u}])\|$.

We consider the following set of assumptions :

(A2) $f: J_a \times J_b \times \mathrm{IR} \to \mathrm{IR}^+ - \{0\}$, and $\dfrac{\varphi(x)}{f(x,0,\varphi(x))} + \dfrac{\psi(x)}{f(0,y,\psi(y))} \geq \dfrac{\varphi(0)}{f(0,0,\varphi(0))}$ for all $(x,y) \in J_a \times J_b$.

(A3) The functions $f(x,y,z)$ is nondecreasing in z almost everywhere for $(x,y) \in J_a \times J_b$.

(B3) The function $k(x,y,t,s,z)$ is nondecreasing in z for all $x,t \in J_a$ and $y,s \in J_b$.

(B4) The mapping $g: J_a \times J_b \times \mathrm{IR} \times \mathrm{IR} \to \mathrm{IR}^+$ is L^1-Caratheodory.

(B5) The function $g(x,y,w,z)$ is nondecreasing in w and z almost everywhere for $(x,y) \in J_a \times J_b$.

(B6) The HIDE (1) has a lower solution u and an upper solution v on $J_a \times J_b$ with $u \leq v$.

Remark 4.1

Assume that (B3), (B4) and (B5) hold. Define a function $h: J_a \times J_b \to \mathrm{IR}^+$ by

$$h(x,y) = g\left(x, y, v(x,y), \int_0^x \int_0^y k(x,y,t,s,v(t,s))\,ds\,dt\right), \forall (x,y) \in J_a \times J_b.$$

Then h is Lebesgue integrable and

$$\left| g\left(x, y, z(x,y), \int_0^x \int_0^y k(x,y,t,s,z(t,s))\,ds\,dt\right)\right| \leq h(x,y), a.e.\,(x,y) \in J_a \times J_b$$

for all $z \in [u,v]$.

Theorem 4.2

Suppose that the assumptions (A1)-(A4) and (B0), (B3)-(B-6) hold. furthermore, if

$$\|\ell\|_\infty \left(\|z_0\|_\infty + \|h\|_{L1}\right) < 1, \tag{10}$$

where h is given in Remark 4.1, then the HIDE (1) has a minimal and a maximal positive solution on $J_a \times J_b$.

Proof.

Now HIDE (1) is equivalent to FIE (3) $J_a \times J_b$. Let $X = C(J_a \times J_b, IR)$. Define two operators A and B on X by (4) and (5) respectively. Then FIE (3) is transformed into an operator equation $Au(x,y)\,Bu(x,y) = u(x,y)$ in Banach algebra X. Notice, hypotheses (A2) and (B4) imply that $A,B\colon [u,v] \to K$. Since the cone K in X is normal, $[u,v]$ is a norm bounded set in X. Now it is shown, as in the proof of Theorem 3.1, that A is a Lipschitz operator with the Lipschitz constant $\|\ell\|_\infty$ and B is completely continuous operator on $[u,v]$. Again, the hypotheses (A3), (B3) and (B4) together implies that A and B are nondecreasing on $[u,v]$. To see this, let $u_1, u_2 \in [u,v]$ be such that x < y. Then by (A3),

$$Au_1(x,y) = f(x,y,u_1(x,y)) \le f(x,y,u_2(x,y)) = Au_2(x,y)$$

for all $(x,y) \in J_a \times J_b$, and

$$\begin{aligned} Bu_1(x,y) &= z_0(x,y) + \int_0^x \int_0^y g\left(t,s,u_1(t,s), \int_0^t \int_0^s k(t,s,\tau,\eta,u_1(\tau,\eta))\,d\eta\,d\tau\right) ds\,dt \\ &\le z_0(x,y) + \int_0^x \int_0^y g\left(t,s,u_2(t,s), \int_0^t \int_0^s k(t,s,\tau,\eta,u_2(\tau,\eta))\,d\eta\,d\tau\right) ds\,dt \\ &= Bu_2(x,y) \end{aligned}$$

for all $(x,y) \in J_a \times J_b$ So A and B are condecreasing operators on $[u,v]$. Again Lemma 4.1 and hypothesis (B5) together implies that

$$\begin{aligned} u(x,y) &= [f(x,y,u(x,y))] \\ &\quad \times \left(z_0(x,y) + \int_0^x \int_0^y g\left(t,s,u(t,s), \int_0^t \int_0^s k(t,s,\tau,\eta,u(\tau,\eta))\,d\eta\,d\tau\right) ds\,dt \right) \\ &\le [f(x,y,z(x,y))] \\ &\quad \times \left(z_0(x,y) + \int_0^x \int_0^y g\left(t,s,z(t,s), \int_0^t \int_0^s k(t,s,\tau,\eta,z(\tau,\eta))\,d\eta\,d\tau\right) ds\,dt \right) \\ &\le [f(x,y,v(x,y))] \\ &\quad \times \left(z_0(x,y) + \int_0^x \int_0^y g\left(t,s,v(t,s), \int_0^t \int_0^s k(t,s,\tau,\eta,v(\tau,\eta))\,d\eta\,d\tau\right) ds\,dt \right) \\ &\le v(x,y), \end{aligned}$$

for all $(x,y) \in J_a \times J_b$ and $z \in [u,v]$. As a result, we have

$$u(x,y) \leq Az(x,y)Bz(x,y) \leq v(x,y)$$

for all $(x,y) \in J_a \times J_b$ and $z \in [u,v]$. Hence, $Az\, Bz \in [u,v]$ for all $z \in [u,v]$. Again

$$\begin{aligned} M &= \|B([u,v])\| \\ &= \sup\{\|Bz\| : z \in [u,v]\} \\ &\leq \sup_{(x,y)\in J_a\times J_b} |z_0(x,y)| \\ &\quad + \sup_{z\in[u,v]}\left\{ \sup_{(x,y)\in J_a\times J_b} \int_0^x\int_0^y \left| g\left(t,s,v(t,s),\int_0^t\int_0^s k(t,s,\tau,\eta,v(\tau,\eta))d\eta d\tau\right)\right| ds\, dt \right\} \\ &\leq \|z_0\|_\infty + \int_0^a\int_0^b h(t,s)ds\, dt \\ &\leq \|z_0\|_\infty + \|h\|_{L^1}. \end{aligned}$$

Since

$$kM \leq \|\ell\|_\infty \left(\|z_0\|_\infty + \|h\|_{L^1}\right) < 1,$$

We know apply Theorem 4.1 to FIE (3) to yield that the HIDE (1) has a minimal and a maximal positive solution on $J_a \times J_b$. This completes the proof.

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SPHERICAL SHOCK WAVES IN A SELF-GRAVITATING NON-IDEAL GAS WITH RADIATION-HEAT FLUX

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ABSTRACT

In the present work, we analyse the propagation of a spherical shock wave in a non-ideal gas, taking into account the effects of radiation-heat flux. Radiation pressure and radiation energy are assumed to be negligible. The gas is supposed to be optically thick and self-gravitating and the initial density to be varying and obeying a power law. The shok is assumed to be transparent for radiation and the counter pressure is taken into account. Using the approximate method of Whitham, an ordinary differential equation is obtained which determine the shock velocity and the shock-Mach number. Effects of the radiation, the non-idealness of the gas and the inhomogeneity of the initial medium on the shock velocity and the shock-Mach number are obtained.

Keywords: Shock waves, Self-Gravitating Gas, Non-Ideal Gas, Radiation-Hear Flux, Whitham's Method

Subject Classification : PACS. 47.40, -x Compressible Flows; shock and detonation phenomena. 47.70 Mc Radiation gas dynamics.

INTRODUCTION

The problem of propagation of shock waves in a nonhomogeneous medium is of great interest in exploring the effects of explosion in stars and atmosphere of earth and in several other branches of engineering and science. Sakurai [17], Rogers [14], Zel'dovich and Raizer [29], Purohit [11], Rosenau and Frankenthal [16], Vishwakarma, Srivastava and Kumar [22], Vishwakarma and Yadav [25], Vishwakarma and Vishwakarma [23], and many others have discussed the propagation of shock waves in a conducting or non-conducting gas with varying density by self-similarity methods. Bhatnagar and Sachdev [2], and Singh and Yadav [18] have studied the propagation of spherical shock waves in a radiative, self-gravitating and non-conducting gas with decreasing density. They used Whitham's [27] rule to obtain the shock velocity and the flow-variables just behind the shock. In all of these works, the medium is assumed to be a gas obeying the equation of state of a perfect gas.

When the flow takes place at extreme conditions, the assumption that the gas is ideal is no more valid. Anisimov and Spiner [1] have taken an equation of state for low density non-ideal gases in a simplified form, and investigated the effect of parameter for non-idealness on the problem of a point explosion. Ranga Rao and Purohit [12] and Ojha [8] have also studied the propagation of shock waves in gases with the above equation of state. In the present work,

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we analyse the propagation of a spherical shock wave in a non-ideal gas with the equation of state given by Anisimov and Spiner [1], taking into account the effects of radiation-heat flux. Radiation pressure and radiation energy are assumed to be negligible. The gas is supposed to be optically thick [3] and self-gravitating and the initial density to be varying and obeying a power law. The shock is assumed to be transparent for radiation [5,26] and the counter pressure is taken into account. As the similarity solutions does not exist for a non-ideal gas with variable initial density [20] we study the present problem by using the approximate method of Whitham (Whitham's rule [27]). Although the Whitham's rule is approximate, it agrees well with exact solutions and with experimental results [6]. Effects of the radiation, the non-idealness of the gas and the inhomogeneity of the initial medium on the shock velocity and the shock-Mach number are obtained.

BASIC EQUATIONS AND BOUNDARY CONDITIONS

The medium under consideration is a non-ideal gas whose equation of state is borrowed from the statistical physics [7], and simplified by Anisimov and Spiner [1] in the form

$$p = \Gamma \rho T\,(1 + b\rho), \tag{2.1}$$

where $b(<< 1)$ is internal volume of the molecules, and Γ, ρ, p, and T are the gas constant, the density, the pressure and the temperature, respectively. Similar equations of state have been used by Wu and Roberts [28] and Roberts and Wu [13] to study the shock wave theory of sonoluminescence and by Vishwakarma and Pandey [21] to study the convergence of cylindrical shock waves in presence of an axial magnetic field.

The internal energy e per unit mass of the non-ideal gas is given by

$$e = \frac{p}{\rho(\gamma-1)\,(1+b\rho)} = \frac{p(1-b\rho)}{\rho(\gamma-1)}, \tag{2.2}$$

which implies that $C_p - C_v = \Gamma\left(1 + \frac{b^2\rho^2}{1+2b\rho}\right) \cong \Gamma,$ (2.3)

neglecting the term $b^2\rho^2$. Here C_p and C_v are the specific heats of the gas at the constant pressure and constant volume respectively, and γ is the ratio of C_p and C_v.

If we assume that the gas is inviscid, non-heat conducting, self-gravitating and radiating, the fundamental equations governing the spherically symmetric flow are [18,19])

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{2\rho u}{r} = 0, \tag{2.4}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{Gm}{r^2} = 0, \tag{2.5}$$

$$\frac{\partial e}{\partial t} + u\frac{\partial}{\partial r} + p\left[\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) + u\frac{\partial}{\partial r}\left(\frac{1}{\rho}\right)\right] + \frac{1}{\rho r^2}\frac{\partial}{\partial r}(r^2 F) = 0, \tag{2.6}$$

$$\frac{\partial m}{\partial r} - 4\pi\rho r^2 = 0\,, \tag{2.7}$$

where m, G u, F r, and t denote the mass of the gas within the radius r, the gravitational constant, the fluid velocity, the radiation heat-flux, the distance from the point of symmetry and the time, respectively. Here, the radiation pressure and the radiation energy are assumed to be negligible.

Assuming local thermodynamic equilibrium and a diffusion model for an optically thick grey gas [10], the differential approximation of the radiation transport equation can be written in the following form

$$F = \frac{c\mu}{3}\frac{\partial}{\partial r}(\sigma T^4)\,, \tag{2.8}$$

where $\frac{1}{4}\sigma c$ is the Stefan-Boltzmann constant, c the velocity of light and μ the Rosseland mean free path for radiation; which is a function of density and temperature.

Following Wang [26], we take

$$\mu = \mu_0 \rho^{\alpha} T^{\beta}, \tag{2.9}$$

where μ_0, α and β are constants.

We assume that the medium in which an explosive shock wave caused by an energy release, is propagating, is inhomogeneous and the density vary inversely as some power of distance [9,15,25],

i.e. $$\rho_0 = \rho_c r^{-w}, \tag{2.10}$$

where ρ_c and w are constants.

Since the fluid originally is in hydrostatic equilibrium, we have

$$m_0 = \frac{4\pi\rho_c}{3-w} r^{3-w}, \tag{2.11}$$

$$p_0 = \frac{2\pi G\rho_c^2}{(3-w)(w-1)} r^{2(1-w)}, \tag{2.12}$$

$$F_0 = -\frac{4c\sigma\mu_0}{3\Gamma^{\beta+4}} \rho_c^{\alpha-\beta-4}\, p_c^{\beta+4}\, R^{2\beta-w(\alpha+\beta+4)+7} \frac{(2-w+2\delta r^{-w})}{(1+\delta r^{-w})^{\beta+5}}, \tag{2.13}$$

where $\delta = b\rho_c$ is the parameter of non-idealness of the gas.

Equation (2.12) may be written as

$$P_0 = p_c r^{2(1-w)} \tag{2.14}$$

where $$p_c = \frac{2\pi G\rho_c^2}{(3-w)(w-1)}, 1 < w < 3. \tag{2.15}$$

The frozen speed of sound (the speed of sound when $b = 0$) a_f and the speed of sound in the non-ideal gas a_0, in the undisturbed state, are given by

$$a_f^2 = \frac{\gamma p_0}{\rho_0}, \tag{2.16}$$

and $$a_0^2 = \frac{\gamma p_0}{\rho_0}\left(\frac{1+2b\rho_0}{1+b\rho_0}\right). \tag{2.17}$$

Therefore $$a_f = \left(\frac{\gamma p_c}{\rho_c}\right)^{\frac{1}{2}} r^{\frac{2-w}{2}}, \tag{2.18}$$

and $$a_0 = a_f \left(\frac{1+2\delta r^{-w}}{1+\delta r^{-w}} \right)^{\frac{1}{2}}. \qquad (2.19)$$

The conservation conditions across the shock front may be wri tten as

$$\rho_2 (U - u_2) = \rho_1 U = m_s \,(say), \qquad (2.20a)$$

$$p_2 - p_1 = m_s u_2, \qquad (2.20b)$$

$$e_2 + \frac{p_2}{\rho_2} + \frac{1}{2}(U - u_2)^2 - \frac{F_2}{m_s} = e_1 + \frac{p_1}{\rho_1} + \frac{1}{2}U^2 - \frac{F_1}{m_s}, \qquad (2.20c)$$

$$m_2 = m_1, \qquad (2.20d)$$

where subscripts "1" and "2" denote the states immediately ahead and behind the shock front and U is the shock velocity. The transition region within the shock wave is taken to be transparent to the radiative-flux [5,26] so that

$$F_2 = F_1 \qquad (2.20e)$$

and the jump conditions across the front reduce to those of a nonradiative shock passing into a non-ideal gas at rest.

The jump conditions may, therefore, be written as

$$u_2 = \frac{2 f(R,M) a_f M}{\gamma + 1}, \qquad (2.21)$$

$$\rho_2 = \frac{\rho_1 (\gamma + 1)}{\gamma + 1 - 2 f(R,M)}, \qquad (2.22)$$

$$p_2 = \frac{p_1}{\gamma + 1}\left[2\gamma M^2 (1 - \delta R^{-w}) - \gamma + 1 \right], \qquad (2.23)$$

where $f(R, M) = (1 - \delta R^{-w} - M^{-2})$ (2.24)

and $$M^2 = \frac{U^2}{a_{f1}^2}.$$

Here R is the shock radius and M the shock-Mach number referred to sound speed a_{f1}.

SOLUTION OF THE PROBLEM

Now, we apply Whitham's rule to get an expression for shock velocity. The equation of motion along the positive characteristic

$\frac{dr}{dt} = u + a$ *is*

$$dp + \rho a du + \left[\frac{\rho a G m}{r^2} + \frac{2\rho u a^2}{r} + \frac{(\gamma-1)(1+b\rho)}{r^2} \frac{\partial}{\partial r}(r^2 F) \right] \frac{dr}{u+a} = 0, \tag{3.1}$$

where $a^2 = \frac{\gamma p}{\rho}\left(\frac{1+2b\rho}{1+b\rho}\right)$.

For diverging shocks, Whitham's rule is to apply the characteristic equation (valid along the positive characteristic) to the flow quantities just behind the shock front. Therefore, substituting the values of u_2, p_2, ρ_2, F_2 and m_2 from the equations (2.21), (2.22), (2.23) and (2.20d, e) in the equation (3.1), we get after some simplifications

$$R\frac{dM}{dR}\left[2\gamma M g + \eta\gamma(f + 2M^2)\right] + \{2\gamma g M^2 - \gamma + 1\}(1 - w) + \gamma\delta w M^2 R^{-w}$$

$$+\left(\frac{2-w}{2}\right)\gamma\eta f M + \gamma\delta w\eta M R^{-w} + \Big[(w-1)(\gamma+1)^2\eta + 2\gamma M f \eta^2(\gamma+1-2f)$$

$$-\frac{2N(\gamma-1)(\gamma+1)^2\gamma^{-\frac{1}{2}}\Gamma^{-\beta-\frac{5}{2}}\rho_c^{\alpha-\beta-\frac{7}{2}}P_c^{\beta+\frac{5}{2}}\xi R^{2(2-\beta)-w\left(\alpha+\beta+\frac{3}{2}\right)}}{(1+\delta R^{-w})^{\beta+5}}$$

$$\times\left\{-2w\delta R^{-w} + \left(\frac{8-5w}{2}\right)(2 - w + 2\delta R^{-w}) + \frac{(\beta+5)(2-w)w\delta R^{-w}}{(1+\delta R^{-w})}\right\}\Big]$$

$$\times\left[2fM + (\gamma+1-2f)\eta\right]^{-1} = 0, \tag{3.2}$$

where $g = (1 - \delta R^{-w})$,

$$\eta(R,M) = \left[\frac{(2\gamma g M^2 - \gamma + 1)\{(\gamma+1)(1+2\delta R^{-w}) - 2f\}}{(\gamma+1-2f)\{(\gamma+1)(1+\delta R^{-w}) - 2f\}}\right]^{\frac{1}{2}},$$

$$\xi(R,M) = \left[\frac{(\gamma+1)(1+\delta R^{-w}) - 2f}{(\gamma+1-2f)}\right],$$

and $N = \dfrac{c\sigma\mu_0}{\Gamma^{3/2}}$ is the radiation parameter.

For the removal of dimensional constants ρ_c and p_c, we assume $\alpha = 1$ *and* $\beta = -\dfrac{5}{2}$; and after some simplifications, we get

$$\frac{dM}{dR} + \frac{1}{R}\left[\frac{K_2}{K_1} + \frac{K_4}{K_1 K_3}\right] - \frac{K_5}{K_1 K_3}\frac{1}{R^2} = 0, \tag{3.3}$$

where

$$K_1(R,M) = 2\gamma g M + \gamma\eta(f + 2M^{-2}),$$

$$K_2(R,M) = \{2\gamma g M^2 - \gamma + 1\}(1-w) + \gamma w \delta M^2 R^{-w} + \frac{2-w}{2}\gamma\eta f M + \gamma\eta w M \delta R^{-w},$$

$$K_3(R,M) = 2fM + (\gamma+1-2f),$$

$$K_4(R,M) = \eta(w-1)(\gamma+1)^2 + 2\gamma M f \eta^2(\gamma+1-2f),$$

$$K_5(R,M) = \frac{2N(\gamma-1)(\gamma+1)^2\gamma^{-\frac{1}{2}}\xi}{(1+\delta R^{-w})^{\frac{5}{2}}}\left[-2w\delta R^{-w} + \left(\frac{8-5w}{2}\right)(2-w+2\delta R^{-w}) + \frac{5w(2-w)\delta R^{-w}}{2(1+\delta R^{-w})}\right].$$

The expressions for the effective shock-Mach number M_e and the shock velocity U in terms of the shock-Mach number M and the shock radius R can be written as

$$M_e = M\left[\frac{1+\delta R^{-w}}{1+2\delta R^{-w}}\right]^{\frac{1}{2}}, \tag{3.4}$$

and

$$\frac{U}{\left(\frac{\gamma p_c}{\rho_c}\right)^{\frac{1}{2}}} = MR^{\frac{2-w}{2}}. \tag{3.5}$$

Equations (3.3), (3.4) and (3.5) give the shock-Mach number, the effective shock-Mach number and the shock velocity as functions of R.

RESULTS AND DISCUSSION

Values of the shock-Mach number M, the effective shock-Mach number M_e and the shock velocity $U\Big/\left(\frac{rp_c}{p_c}\right)^{1/2}$ are calculated from equations (3.3), (3.4) and (3.5) by numerical integration. For numerical integration we have taken $\gamma = 1.4$; $w = 1.1, 1.2$; $N = 0, 1, 5$; $\delta = 0, 0.025$. [4,12,15] and the initial conditions as $M = 4$ at $R = 0.05$ (c.f. [18, 24]). The particular values $\delta = 0$ and $N = 0$ are, respectively, associated with the case of a perfact gas and the radiation free case.

Figure 1 shows the variation of the shock-Mach number M with the shock radius R. It is found that M decreases as R increases, and this decrease is very rapid in the beginning, becomes slower there after and continues until M is reduced to unity. The effective shock-Mach number M_e displays similar behaviour as M (see figures 1 and 2). It means that the shock decays very fast after its formation, and reduces into a sound wave. Figure 3 shows that, as R increases, the reduced shock velocity $U\Big/\left(\frac{rp_c}{p_c}\right)^{1/2}$ increases fast, after a decrease for a very short time; whereas the shock strength (M_e) decreases (figure 2). In fact, as R increases the reduced sound speed $a_1 \Big/ \left\{\left(\frac{rp_c}{p_c}\right)^{1/2}\right\} = \left(\frac{1+2\delta R^{-w}}{1+\delta R^{-w}}\right) R^{(2-w)/2}$ increases

faster than the reduced shock velocity $U\Big/\left(\dfrac{rp_c}{\rho_c}\right)^{\frac{1}{2}}$ and this results in a decrease in the shock strength M_e.

Figures 1, 2 and 3 show that the effects of an increase in the radiation parameter N are

(i) to increase the shock-Mach number M and the effective shock-Mach number M_e, and

(ii) to increase the shock velocity $U\Big/\left(\dfrac{rp_c}{\rho_c}\right)^{\frac{1}{2}}$

Thus the effects of radiation-heat flux on the shock wave are to increase its velocity and strength. The effects of the non-idealness of the gas are to decrease the shock-Mach number, the effective shock-Mach number and the shock velocity (figure 1, 2 and 3). Comparing the curves 2 and 4 with the curve 10 and 12, we see that these effects are significant when the gas is non-radiative ($N = 0$). Thus the non-idealness of the gas has decaying effect on the shock wave, and the presence of radiation reduces its effect.

An increase in the value of the inhomogeneity index w is to increase the shock-Mach number and the effective shock-Mach number in the case of perfect gas ($\delta = 0$), and to decrease these quantities in the case of non-ideal gas ($\delta = 0.025$).

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HERMITIAN TYPE OPERATORS OF SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT

The operator theory is studies which is close to some finer points of Quantum Mechanics and Mathematical Chemistry. The paper proves the second order differential operator to be Hermitian under the specific conditions.

INTRODUCTION

Operators theory has received inspiration from differential and integral equations. Recent advances in Volterra integral and functional equations have enabled one to reflect more closely on some of the finer points of Quantum Mechanics and Mathematical Chemistry. Our objective is to do of new interest some of the useful broader questions in Physics and Chemistry. The simplest form of an operator is differential operator $D\,u(t) = u'(t)$. The differencial operator $\Delta\, u(t) = u(t+1)\text{-}u(t)$ and

Here $D^2\, ut(t) = D[Du(t)] = D[u'(t)] = u''(t)$.

$$\begin{aligned}\Delta^2 u(t) &= \Delta[\Delta u(t)] = \Delta[u(t+1) - u(t)]\\ &= u(t+2) - u(t+1) - u(t+1) + u(t)\\ &= u(t+2) - 2u(t+1) + u(t)\end{aligned}$$

The Integral Operator can be started as $(kf)(x) = \int_a^x f(y)dy / \sqrt{x-y}$

In this case

$$\begin{aligned}k^2 f = k(kf) &= \int_a^x (kf)(y)dy / \sqrt{x-y}\\ &= \int_a^x y\left[\int_a^x f(s)ds / \sqrt{y-s}\right]\left[dy / \sqrt{x-y}\right]\end{aligned}$$

Generally operators act on functions. Some well-known operators in Quantum Mechanics are as follows.

(1) The operator corresponding to linear momentum

$$p = -i\hbar\nabla$$

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(2) The operator corresponding to angular momentum

$$l = -i\hbar r \times \nabla$$

(3) The operator of energy equation.

$$(1/2m)p^2 + V = -(h^2/2m)\nabla^2 + V$$

(4) The Hamiltonian operator is given by

$$H\psi_\lambda = -(h^2/2m)\nabla^2\psi_\lambda + V(x,y,z)\psi_\lambda = E_\lambda\psi_\lambda$$

It can be written in more general form

$$\nabla^2\psi_\lambda + 2m/h^2(E_\lambda - V)\psi_\lambda = 0$$

This is known as SchrÖdinger operator bearing his name. This can be expressed in the form $H(a,v) = (p-\alpha)^2 + V$ acting in the Hilbert space $H = L^2(R^3)$ where $p = -i\nabla$ and $\alpha : R^3 \longrightarrow R^3$, $V : R^3 \longrightarrow R$ are functions, which are called vector and scalar potential respectively. This is oftenly used in Quantum Mechanics. In fact this operator is a foundation of Quantum Mechanics. The following operator can also said to be basic foundation of quantum theory, which is called as Dirac operator [2], defined by $H_0 = \alpha p + \beta.m$, which describes a relativistic electron with mass $m > 0$, the dot means the usual scalar product in R^3 and $p = -i\nabla$ is the momentum observable and $\beta = \alpha_0, \alpha_1, \alpha_1, \alpha_2, \alpha_3$ are Dirac matrices satisfying the anti-commutation relations $\alpha_j\,\alpha_k + \alpha_k\,\alpha_k = 2\delta_{jk}$, where δ_{jk} is Kroneckers delta.

In Quantum Mechanics the operators such as momentum, angular momentum, energy are all self adjoint operators and known as observables. This paper is subjected to differential operators of the form $L(u) = u'' + Dg(u')$ where $Dg(.)$ is the derivative of the product g (.).

The inspiration of this operator goes to the second order differential equations given by Margenau and Murphy [3]. We are going to prove that the operator $L(u)$ is a Hermitian with respect to the functions under the specific conditions. The importance of the Hermitian operators has been significantly used in Quantum Mechanics and Mathematical Physics. Our consideration is limited to the following second order differential equations.

$$(1 + g(x))y'' + g'(x)y' + h(x)y = 0 \tag{1.1}$$

$$xy'' + y' + h(x)y = 0 \tag{1.2}$$

and $$y'' + [k + h(x)]y = 0 \tag{1.3}$$

and h is a periodic function of x that is $h(x + \ell) = h(x)$. This is known as SchrÖdinger equation.

Let, us suppose that $\phi(x)$ and $\psi(x)$ are the corresponding permissible solutions of the equations (1.1) to (1.3). Our interest in this paper is to prove at the operators of Eqns. (1.1) to (1.3) for solutions $\phi(x)$ and $\psi(x)$ respectively are Hermitian in the sense defined in section 2.

If the unit cofficient term y'' is absent in (1.1) and $h(x)$ is replaced by $-h(x)$ then it becomes a Strum Liouvilles equation

$$g(x)\,y'' + g'(x)y' - h(x)\,y = 0 \tag{1.4}$$

In the sequance we prove the uniqueness and stability of the solutions of second order differential equation

$$y'' = f(x,y,y') \tag{1.5}$$

with $y(0) = y_0$ and $y'(0) = y_0'$

HERMITIAN TYPE OPERATORS

There exists function in Mathematics and natural sciences, the integral of which vanishes at the extreme end points. For the trignometric function such as *cos x* and *sin x*, the integrals of *cos x* vanish at the end points 0 to π. The integrals of $\sin x$ vanish at $-\pi/2$ and $\pi/2$. The interesting example where the integeral part vanishes at both upper and lower limits is the Gamma function.

We call that $\Gamma_{(z)} = \lim_{n\to\infty} F(z,n)$, where $F(z,n) = \int_0^n (1-t/n)^n t^{z-1} dt$, 'n' stands for a positive integer, and real part of z is taken to be greater than zero. In the above integrand, Let $r = t/n$; this converts F into

$$F(x,n) = n^2 \int_0^n (1-r)^n r^{z-1} dr.$$

Integrating the function under the integral we obtain,

$$\int_0^1 (1-r)^n r^{z-1} dr = [(1-r)^n r^z / z]_0^{\ 1} + \frac{n}{z}\int_0^1 (1-r)^{n-1} r^z dr$$

The integral part vanishes at the both end points. The remainder may again be subjected to a partial integration, in which the integral part is again zero at the end limits, it continuse upto infinitely many numbers. We are interested in this continued integral part zero at the end limits in our investigation. With this simple and beautiful promise in mind, we define the following useful terms:

Definition 2.1

A function ϕ is said to be integral vanishing part at end limits α and β if

$$\phi\psi'\,\Big]_{\alpha} = \psi\phi'\,\Big]_{\beta} = 0$$

Remark : If the differential equation is of the form $f(x)y'' + g(x)y' + h(x)\,y = 0$ (2.1)
then above condition can be extended to the following general format:

$$f\,\phi\psi'\Big]_{\alpha} = \psi\phi'\Big]_{\beta} = 0$$

On the basis of this condition we define Hermitian operator L with respect to the function satisfying the condition 2.1. Hermitian operators have great importance in quantum theory.

Definition 2.2 :

Let ϕ and ψ be two permissible solutions of equation (1.1). An operator L is said to be Hermitian with respect these functions satisfying condition of definition (2.1) if

$$\int_{\alpha}^{\beta} \phi L(\psi)dx = \int_{\alpha}^{\beta} \psi L(\phi)dx.$$

Let us consider the differential equation (1.1) The differential operator $L(\phi) = \phi'' + Dg(\phi)$ is called self-adjoint operator. We recall that the necessary and sufficient condition for the general second order differential operator $D(y) = fy'' + gy' + hy$ in which f, g, h are functions of independent variable x, to be self-adjoint is that $g = f$. It can also be noted that every second order differential operator $D(y)$ can be made self-adjoint if we multiply it from the left by the factor $\exp\left[\int[(g-f')/f]dx\right]$.

As an example consider the second order differential operator,

$$D(y) = (1-x^2)y'' - 2xy' + \left[\ell(\ell+1) - m^2/(1-x^2)\right]y$$

Here $\quad f(x) = 1 - x^2, \;\; g(x) = -2x$

Hence $\quad \int (g(x) - f'(x)/f(x))dx = \int (-2x + 2x/(1-x^2))dx = 0$

Thus $\quad \exp\left[\int (g - f'/f)dx\right] = 1.$

Multiplying by 1, the operator D(y) in this case become self adjoint. We also note that $f'(x) = -2x = g'(x)$. The necessary condition is also verified for this operator.

Theorem 2.1:

Let D be the first order differential operator. Then the second order differential operator $L(\phi) = \phi'' + Dg(\phi') + h(x)\phi$ is Hermitian, for second order differential equation (1.1).

Proof :

Let ϕ and ψ be two solutions of equation (1.1). We claim that

$$\int_{\alpha}^{\beta} \psi \ L(\phi) dx = \int_{\alpha}^{\beta} \phi L(\psi) \ dx$$

Let us consider,

$$\int_{\alpha}^{\beta} \psi \ L(\phi) dx = \int_{\alpha}^{\beta} \psi[\phi'' + Dg(\phi') + h(x)\phi] dx.$$

$$= \int_{\alpha}^{\beta} [\psi \ \phi'' + \psi g \phi'' + \psi g' \phi' + h(x)\psi \ \phi] dx.$$

$$= \int_{\alpha}^{\beta} \psi\phi'' \, dx + \int_{\alpha}^{\beta} \psi g \phi'' \, dx + \int_{\alpha}^{\beta} \psi g' \phi' \, dx + \int_{\alpha}^{\beta} \psi h \phi dx.$$

on partial integration and keeping last two as they are, we get

$$\int_{\alpha}^{\beta} \psi \ L(\phi) dx = \psi\phi' \Big]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \phi' \psi' \, dx + \psi g \phi' \Big]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \phi' D(\psi g) dx$$

$$+ \int_{\alpha}^{\beta} \psi g' \phi' \, dx + \int_{\alpha}^{\beta} \psi \ h \ \phi \ dx.$$

By definition (2.1) and (2.2) we obtain

$$\int_{\alpha}^{\beta} \psi \ L(\phi) \ dx = \int_{\alpha}^{\beta} \phi \ L(\psi) dx.$$

Now let us turn to equation (1.4).

Theorem 2.2:

Let ϕ and ψ be two solutions of equation (1.4); Then the operator $L(\phi) = (g\phi')' - h\phi$ is Hermitian.

Proof : Let us consider

$$I = \int_{\alpha}^{\beta} \psi L(\phi) dx = + \int_{\alpha}^{\beta} \psi[(g\phi')'] - \int_{\alpha}^{\beta} \psi h \phi dx$$

$$= \int_{\alpha}^{\beta} (\psi g \phi'' + \psi g' \phi') dx - \int_{\alpha}^{\beta} \psi h \phi dx.$$

On partially integrating we obtain;

$$I = \psi g\phi' \Big]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \phi' D(\psi g)dx + \int_{\alpha}^{\beta} \psi g'\phi' dx - \int_{\alpha}^{\beta} \psi h\phi dx.$$

The first term vanishes by the condition. So

$$I = -\int_{\alpha}^{\beta} (\phi'\psi g' + \phi' g\psi')dx + \int_{\alpha}^{\beta} \psi g'\phi' dx - \int_{\alpha}^{\beta} \psi h\phi dx.$$

Further integration yields

$$I = -\phi\psi' g \Big]_{\alpha}^{\beta} + \int_{\alpha}^{\beta} \phi D(\psi' g)dx - \int_{\alpha}^{\beta} \phi'\psi g' dx + \int_{\alpha}^{\beta} \psi g'\phi' dx - \int_{\alpha}^{\beta} \psi h\phi dx.$$

Using definition (2.1) and (2.2) and simplifying we get,

$$I = \int_{\alpha}^{\beta} \phi L(\psi)dx.$$

In fact theorem (2.1) is more general than theorem (2.2). But for the self perfectness, we provide it's proof.

Let us now consider the Legendre's equation $(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$ (2.2)

Where 'ℓ' is a constant. We attempt to find its self adjoint second order differential operator.

Theorem 2.3 :

Let D be the first order differential operator then the second order differential operator $L(\phi)$ i.e. $L(\phi) = \phi'' - Dg(\phi) + \ell(\ell+1)\phi$ is Hermitian for the equation (2.2)

Proof:

Let ϕ and ψ be two solutions of equation (2.2). We claim that

$$\int_{\alpha}^{\beta} \psi L(\phi)dx = \int_{\alpha}^{\beta} \phi L(\psi)dx$$

Let us Consider

$$\int_\alpha^\beta \psi L(\phi)dx = \int_\alpha^\beta \psi\left[\phi'' - Dg(\phi')\right]dx + \int_\alpha^\beta \phi\ell(\ell+1)\psi dx$$

$$= \int_\alpha^\beta \left[\psi\phi'' - \psi g\phi'' - \psi g'\phi'\right]dx + \int_\alpha^\beta \phi\ell(\ell+1)\psi dx$$

$$\int_\alpha^\beta \psi L(\phi)dx = \psi\phi\Big]_\alpha^\beta - \int_\alpha^\beta \phi'\psi'\,dx - \psi g\phi'\Big]_\alpha^\beta + \int_\alpha^\beta \phi' D(\psi g)dx$$

$$- \int_\alpha^\beta \psi g'\phi'\,dx + \int_\alpha^\beta \phi\ell(\ell+1)\psi dx.$$

From definition (2.1) and (2.2) and on simplification we get result,

$$\int_\alpha^\beta \psi L(\phi)dx = \int_\alpha^\beta \phi L(\psi)dx.$$

The equation (1.2) is Fuch equation for $h(x) = -1/x$. We now find its second order operator and prove that it is also Hermitian.

Theorem 2.4:

Let D be the first order differential operator, then

$L(\phi) = Dx(\phi') + h(x)\,Ø$ is Hermitian for the equation (1.2)

Proof:

Let ϕ and ψ be two solutions of equation (1.2), then

$$\int_\alpha^\beta \psi L(\phi)dx = \int_\alpha^\beta \psi\left[Dx(\phi')\right]dx + \int_\alpha^\beta \psi h\phi dx.$$

$$= \int_\alpha^\beta \left[\psi x\phi'' + \psi\phi'\right]dx + \int_\alpha^\beta \psi h\phi dx$$

$$= x\psi\phi\Big]_\alpha^\beta - \int_\alpha^\beta \phi' D(x\psi)dx + \int_\alpha^\beta \psi\phi'\,dx + \int_\alpha^\beta \psi h\phi dx$$

From definition (2.1) and (2.2) and on simplification we get result,

$$\int_{\alpha}^{\beta} \psi L(\phi)dx = \int_{\alpha} \phi L(\psi)dx.$$

Finally we deduce the operator of the equation (1.3) which is called SchrÖdinger equation and prove that it is also Hermitian. This equation has of the form $\phi = e^{ikx}\psi(x)$ where ψ is also periodic; $\psi(x+\ell)=\psi(x)$. This very important property is oftenly used in Quantum Mechanics.

Theorem (2.5) :

The operator $L(\phi) = \phi'' + [k + h(x)]\,\phi$ is Hermitian for the second order differential equation (1.3).

Proof :

The proof is straight forward, let us consider,

$$\int_{\alpha}^{\beta} \psi L(\phi)dx = +\int_{\alpha}^{\beta} \psi\left[\phi'' + (k+h(x))\phi\right]dx$$

$$= \int_{\alpha}^{\beta} \psi\ \phi''dx + \int_{\alpha}^{\beta} \psi\left[(k+h(x))\phi\right]dx$$

Partially integrating the first factor twice, we obtain,

$$\int_{\alpha}^{\beta} \psi L(\phi)dx = \psi\phi'\Big]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \phi'\psi'\,dx + \int_{\alpha}^{\beta} \psi(k+h(x))\phi dx$$

$$= -\psi'\phi\Big]_{\alpha}^{\beta} + \int_{\alpha}^{\beta} \phi\psi''\,dx + \int_{\alpha}^{\beta} \psi(k+h(x))\phi dx$$

From defination 2.1 and defination 2.2 and on simplification, we get,

$$\int_{\alpha}^{\beta} \psi L(\phi)dx = \int_{\alpha}^{\beta} \phi L(\psi)dx.$$

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